MthE 237
Lecture 09
Sept. 29, 2017

Topics:
- Problems that give rise to linear equations
- Definition of the Wronskian

Reminders from last time:

A differential equation of the form

$$(L)\quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)\,y = F(t)$$

is called linear. When $F = 0$, it is called homogeneous:

$$(H)\quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)\,y = 0.$$

We have a basic Existence and Uniqueness Theorem:

Theorem. If $a_{r-1}, \ldots, a_1, a_0$ in (H) are continuous on an open interval $I \subset \mathbb{R}$, then there exists a unique solution $\varphi : I \to \mathbb{R}$ of (H) satisfying the initial conditions

$$\varphi(t_0) = \varphi_0^{(0)}, \quad \frac{d\varphi}{dt}(t_0) = \varphi_0^{(1)}, \ldots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = \varphi_0^{(r-1)}$$

(here $t_0 \in I$, and $\varphi_0^{(0)}, \ldots, \varphi_0^{(r-1)}$ are real numbers)

As a consequence, we have shown

Theorem (Principle of Superposition) The set of solutions of (H) has the structure of an r-dimensional vector space; the map

$$\Psi_{t_0} : \varphi \longmapsto \left(\varphi(t_0), \frac{d\varphi}{dt}(t_0), \ldots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) \right)$$

is an isomorphism for every $t_0 \in I$.

<u>Remarks.</u> 1) The Principle of Superposition is the following statement: If φ_1 and φ_2 are solutions of (H), and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1\varphi_1 + \alpha_2\varphi_2$ is again a solution of (H). In other words, the set of solutions of (H) is a vector space; given two solutions, we may superpose them.

2) The somewhat vague "rule of thumb" from the beginning, namely that solutions to differential equations of order r tend to come in families that depend on r parameters, is made completely precise in the above theorem. There is an r-dimensional vector space of solutions to (H), and we can think of the inverse $\Psi_{t_0}^{-1}$ as "picking out a solution using initial conditions":

$$\Psi_{t_0}^{-1} : \underbrace{\left(\varphi_0^{(0)}, \varphi_0^{(1)}, \dots, \varphi_0^{(r-1)}\right)}_{\text{chosen initial conditions at } t_0} \longmapsto \varphi$$

Solution satisfying

$$\varphi(t_0) = \varphi_0^{(0)}, \quad \frac{d\varphi}{dt}(t_0) = \varphi_0^{(1)}, \quad \dots, \quad \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = \varphi_0^{(r-1)}$$

(As a reminder, every isomorphism $L : V \to W$ has an <u>inverse</u> map, a linear map $L^{-1} : W \to V$ that satisfies $L \circ L^{-1} = Id_W$ and $L^{-1} \circ L = Id_V$.)

Some more terminology:

<u>Def.</u> If $a_{r-1}(t), \dots, a_1(t), a_0(t)$ are constant functions, then (H) is said to have <u>constant coefficients</u>; otherwise, (H) is said to have <u>time-varying coefficients</u>.

<u>Def.</u> A basis of the space of solutions of (H) is called a <u>fundamental set of solutions</u>.

— § —

Here are a few examples of problems that give rise to linear differential equations.

First order

- Compound Interest

let $y(t)$ = Amount of funds in a bank account at time t.

y is said to accrue compound interest at a constant rate of $100 \cdot k$ percent per annum if it satisfies the differential equation

$$\frac{dy}{dt} = ky.$$

The rate may depend on time. For example,

$$\frac{dy}{dt} = \frac{\left(1 + \frac{\cos(t)}{10}\right)}{100} y$$

models a rate that has a minor periodic oscillation about 1 percent/annum.

- Radioactive decay

$y(t)$ = Mass of material undergoing radioactive decay.

$$\frac{dy}{dt} = -ky$$

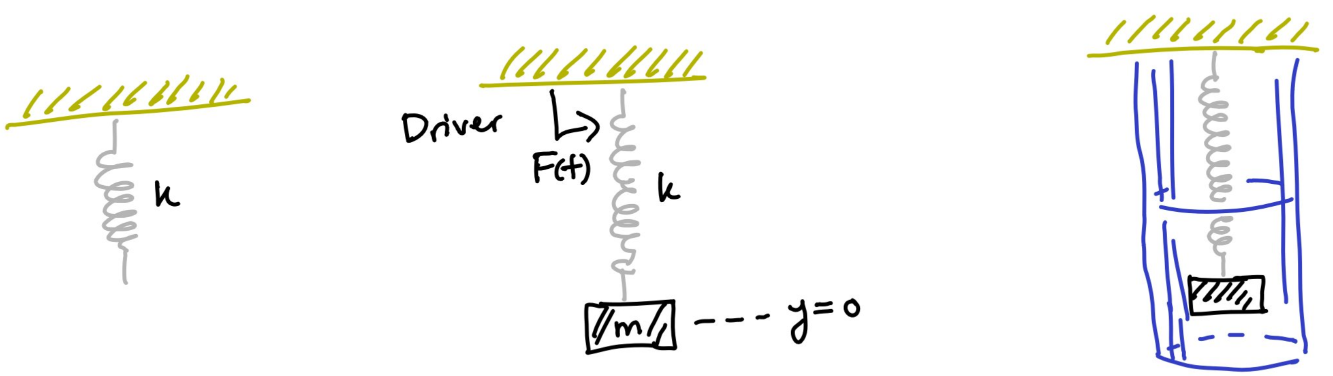
Constant of proportionality typically determined by half-life.

- Numerous other applications

<u>Second order</u>

- Damped harmonic oscillator

Medium with higher damping



Attaching a mass to a spring will stretch the spring to a new equilibrium point to balance the force of gravity on m. Let $y=0$ be the new equilibrium point.

The forces on the mass are

- Restoring force due to the spring

$$-k\,y(t) \quad \text{(Hooke's Law)}$$

Spring constant

- Damping of the medium (or damper)

$$-d\,\dot{y}(t)$$

Reminder: $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$, ...

Damping constant

- Force due to driver or motor

$$F(t).$$

Combining these, Newton's Second Law gives

$$m\ddot{y}(t) = F(t) - d\dot{y}(t) - ky(t), \quad \text{or}$$

$$m\frac{d^2y}{dt^2} + d\frac{dy}{dt} + ky = F(t).$$

In standard form,

$$\frac{d^2y}{dt^2} + \frac{d}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F(t)}{m}.$$

This is a second-order linear differential equation with constant coefficients.

An interesting variant with time-varying coefficients arises if the mass is a function of time. Then Newton's Second Law is

$$F = \underbrace{\dot{p}}_{\text{Momentum}} = \frac{d}{dt}(m\dot{y}) = \dot{m}\dot{y} + m\ddot{y}$$

and the governing equation is

$$m(t)\frac{d^2y}{dt^2} + (\dot{m}(t) + d)\frac{dy}{dt} + ky = F(t).$$

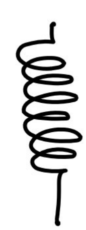
▶ RLC circuit

This is a circuit made up of the following elements:

Resistor

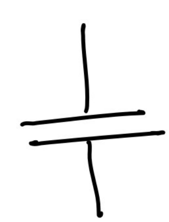
$V = IR$

(Ohm's Law)



Inductor

$V = L\dot{I}$

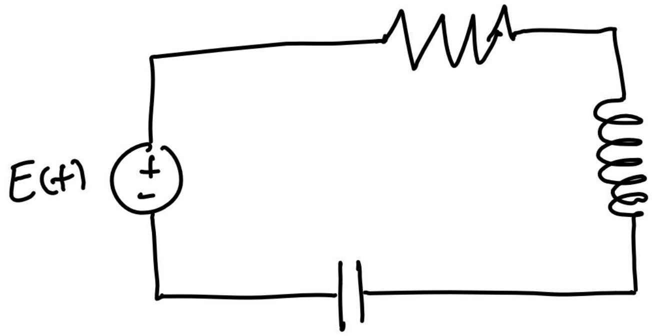


Capacitor

$V = \frac{q}{C}$

Here $q(t) =$ charge

$I(t) =$ current $= \dot{q}(t)$.



By Kirchoff's Voltage Law, the sum of voltage drops as we go around a loop is zero. (This is just conservation of energy).

So $q(t)$ satisfies the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Differentiating, the current $I(t)$ satisfies the differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t).$$

§

Methods of Establishing Linear Independence in $C^r(I, \mathbb{R})$:

Method 1: Proceed directly from the definition of linear independence:

Show that if functions $f_1, \ldots, f_k$ in $C^r(I, \mathbb{R})$ satisfy

$$\alpha_1 f_1 + \cdots + \alpha_k f_k = 0$$

for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, then

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

Example. Look at the equation

$$\frac{d^2y}{dt^2} + y = 0$$

(The Simple Harmonic Oscillator (no damping: $d=0$))

We noticed before that

$\varphi_1(t) = \sin(t)$ and
$\varphi_2(t) = \cos(t)$

are two solutions. If we check that they are linearly independent, we'll have a basis for solutions (since the equation has order 2, the space of solutions is two-dimensional).

We check that $\sin(t)$ and $\cos(t)$ are linearly independent (as elements of $C^r(\mathbb{R}, \mathbb{R})$):

Suppose

$$\alpha_1 \varphi_1 + \alpha_2 \varphi_2 = 0$$

$$\alpha_1 \sin(t) + \alpha_2 \cos(t) = 0 \quad \text{for all } t.$$

At $t = 0$, the equality is

$$0 = \alpha_1 \sin(0) + \alpha_2 \cos(0) = 0 + \alpha_2 \cdot 1$$

At $t = \frac{\pi}{2}$, the equality is

$$0 = \alpha_1 \sin\left(\frac{\pi}{2}\right) + \alpha_2 \cos\left(\frac{\pi}{2}\right) = \alpha_1 \cdot 1 + \alpha_2 \cdot 0$$

Therefore, $\alpha_1 = \alpha_2 = 0$. $\{\sin(t), \cos(t)\}$ is linearly independent.

Aside:

Only for the interested, here is a general version of this computation that does not involve a particular choice of two points.

If we picked another pair of points t_0 and t_1, we would find

$$0 = \alpha_1 \sin(t_0) + \alpha_2 \cos(t_0)$$

$$0 = \alpha_1 \sin(t_1) + \alpha_2 \cos(t_1).$$

This can be written as the matrix equation

$$\begin{pmatrix} \sin(t_0) & \cos(t_0) \\ \sin(t_1) & \cos(t_1) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We compute that

$$\det\begin{pmatrix} \sin(t_0) & \cos(t_0) \\ \sin(t_1) & \cos(t_1) \end{pmatrix} = \sin(t_0)\cos(t_1) - \cos(t_0)\sin(t_1)$$
$$= \sin(t_0 - t_1)$$

If $t_0 - t_1 \neq k\pi$ for any integer k,

then the above determinant is not zero, which implies that $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

If $t_0 - t_1 = k\pi$, we made a particularly poor choice of points, and the above argument would not work.

Method 2: Wronskians

Def. For $f_1, \dots, f_r \in C^{r-1}(I, \mathbb{R})$, their Wronskian is the function

$$W(f_1, \dots, f_r)(t) = \det\begin{pmatrix} f_1 & \cdots & f_r \\ \frac{df_1}{dt} & \cdots & \frac{df_r}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1}f_1}{dt^{r-1}} & \cdots & \frac{d^{r-1}f_r}{dt^{r-1}} \end{pmatrix}$$

with domain I.

<u>Example.</u> $f_1(t) = \sin(t)$ in $C^1(\mathbb{R}, \mathbb{R})$
$f_2(t) = \cos(t)$

$$W(\sin(t), \cos(t))(t) = \det\begin{pmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{pmatrix}$$

$$= -\sin^2(t) - \cos^2(t)$$

$$= -(\sin^2(t) + \cos^2(t))$$

$$= -1 \quad \text{for all } t.$$

We shall return to Wronskians and their properties next class.

Have a good weekend, everyone :)