Mthe 237
Lecture 08
Sept. 27, 2017

Topic: Structure of Solutions of Homogeneous Linear Differential Equations

Goal for today: Prove that the set of solutions of a homogeneous linear differential equation of order r is an r-dimensional vector space

Defs. A differential equation of the form

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F(t)$$

is called linear. It is called homogeneous if $F = 0$. As before, the positive integer r is called the order of the differential equation.

(This use of the term homogeneous is not related to the previous one!)

Examples:

- $\frac{dy}{dt} = kt$ – Not homogeneous
- $\frac{dy}{dt} - ky = 0$ – Homogeneous
- $\frac{d^2y}{dt^2} + y = 0$ – Homog
- $\frac{d^3y}{dt^3} + \tan(t+1)\frac{dy}{dt} + (t^2-1)y = \sqrt{t^4+t^2}$ – Not homog.

Remark: Linear differential equations are linear in $y, \frac{dy}{dt}, \ldots, \frac{d^ry}{dt^r}$, but not t.

For now, we will focus on homogeneous linear differential equations.

The basic theorem (that we will take for granted) is

Theorem [Existence and Uniqueness for Homogeneous Linear Equations]

Consider the differential equation

$$(H) \quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

If the functions $a_{r-1}, \ldots, a_1, a_0$ are continuous on an open interval I, then there exists a unique solution $\varphi: I \to \mathbb{R}$ of (H) satisfying the initial conditions

$$\varphi(t_0) = \varphi_0^{(0)}, \quad \frac{d\varphi}{dt}(t_0) = \varphi_0^{(1)}, \ldots, \quad \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = \varphi_0^{(r-1)},$$

where $t_0 \in I$ and $\varphi_0^{(0)}, \ldots, \varphi_0^{(r-1)} \in \mathbb{R}$.

Remark. The conclusion of the above theorem is existence of a solution over the entire interval I. This is a "global" conclusion, as opposed to the "local" one provided by the previous existence and uniqueness theorem (existence in an interval, possibly very small, about the initial condition).

§ Space of r-continuously differentiable functions

For any non-negative integer r and open interval $I \subset \mathbb{R}$, let $C^r(I,\mathbb{R})$ denote the set of functions $f: I \to \mathbb{R}$ such that

$$f, \frac{df}{dt}, \ldots, \frac{d^r f}{dt^r} \quad \text{all exist and are continuous.}$$

One can check that $C^r(I,\mathbb{R})$ is a real vector space with the operations

$$(f+g)(t) = f(t)+g(t)$$
$$(\alpha f)(t) = \alpha f(t), \quad \alpha \in \mathbb{R}$$

The zero element is the function $f(t)=0$ for all $t \in I$.

The interested student is encouraged to check the vector space axioms for themselves!

The vector space $C^r(I,\mathbb{R})$ is infinite-dimensional (unless $I=\emptyset$...).

Let $Sol(H)$ denote the set of solutions of the differential equation

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t) = 0$$

over an interval I.

(A better rotation would indicate dependence on $a_{r-1}, \dots, a_0$ and I, but we press on...)

Theorem. Suppose that the coefficient functions

$$a_{r-1}, \dots, a_0$$

are continuous over I.

Then $Sol(H)$ is an r-dimensional subspace of $C^r(I, \mathbb{R})$.

Proof. It is clear that $Sol(H)$ is a subset of $C^r(I,\mathbb{R})$.

To check that it is a subspace, we need to check:

1) If $\varphi_1, \varphi_2 \in Sol(H)$, then $\varphi_1 + \varphi_2 \in Sol(H)$

2) If $\varphi \in Sol(H)$, and $\alpha \in \mathbb{R}$, then $\alpha\varphi \in Sol(H)$.

1)
$$\frac{d^r(\varphi_1+\varphi_2)}{dt^r} + a_{r-1}(t)\frac{d^{r-1}(\varphi_1+\varphi_2)}{dt^{r-1}} + \dots + a_1(t)\frac{d(\varphi_1+\varphi_2)}{dt} + a_0(t)(\varphi_1+\varphi_2)$$

$$= \left(\frac{d^r\varphi_1}{dt^r} + \frac{d^r\varphi_2}{dt^r}\right) + a_{r-1}(t)\left(\frac{d^{r-1}\varphi_1}{dt^{r-1}} + \frac{d^{r-1}\varphi_2}{dt^{r-1}}\right) + \dots$$

$$\dots + a_1(t)\left(\frac{d\varphi_1}{dt} + \frac{d\varphi_2}{dt}\right) + a_0(t)(\varphi_1+\varphi_2)$$

$$= \left(\frac{d^r \varphi_1}{dt^r} + a_{r-1}(t) \frac{d^{r-1}\varphi_1}{dt^{r-1}} + \cdots + a_1(t) \frac{d\varphi_1}{dt} + a_0(t)\varphi_1 \right)$$

$$+ \left(\frac{d^r \varphi_2}{dt^r} + a_{r-1}(t) \frac{d^{r-1}\varphi_2}{dt^{r-1}} + \cdots + a_1(t) \frac{d\varphi_2}{dt} + a_0(t)\varphi_2 \right)$$

$$= 0 + 0 = 0$$

$\uparrow$ Because $\varphi_1, \varphi_2 \in Sol(H)$.

2) $$\frac{d^r(\alpha\varphi)}{dt^r} + a_{r-1}(t) \frac{d^{r-1}(\alpha\varphi)}{dt^{r-1}} + \cdots + a_1(t) \frac{d(\alpha\varphi)}{dt} + a_0(t)(\alpha\varphi)$$

$$= \alpha \frac{d^r\varphi}{dt^r} + \alpha a_{r-1}(t) \frac{d^{r-1}\varphi}{dt^{r-1}} + \cdots + \alpha a_1(t) \frac{d\varphi}{dt} + \alpha a_0(t)\varphi$$

$$= \alpha \left(\frac{d^r\varphi}{dt^r} + a_{r-1}(t) \frac{d^{r-1}\varphi}{dt^{r-1}} + \cdots + a_1(t) \frac{d\varphi}{dt} + a_0(t)\varphi \right)$$

$$= \alpha \cdot 0 = 0.$$

$\uparrow$ Because $\varphi \in Sol(H)$

Now, to show that $\mathrm{Sol}(H)$ has dimension r, we construct an isomorphism with $\mathbb{R}^r$:

$$\Psi_{t_0} : \mathrm{Sol}(H) \longrightarrow \mathbb{R}^r$$
$$\varphi \longmapsto \left(\varphi(t_0), \frac{d\varphi}{dt}(t_0), \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) \right),$$

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where t_0 is a fixed element of I.

First, Ψ_{t_0} is a linear map:

$$\Psi_{t_0}(\varphi_1 + \varphi_2)$$
$$= \left((\varphi_1 + \varphi_2)(t_0), \frac{d(\varphi_1+\varphi_2)}{dt}(t_0), \dots, \frac{d^{r-1}(\varphi_1+\varphi_2)}{dt^{r-1}}(t_0) \right)$$
$$= \left(\varphi_1(t_0) + \varphi_2(t_0), \frac{d\varphi_1}{dt}(t_0) + \frac{d\varphi_2}{dt}(t_0), \dots, \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) + \frac{d^{r-1}\varphi_2}{dt^{r-1}}(t_0) \right)$$
$$= \left(\varphi_1(t_0), \frac{d\varphi_1}{dt}(t_0), \dots, \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) \right) + \left(\varphi_2(t_0), \frac{d\varphi_2}{dt}(t_0), \dots, \frac{d^{r-1}\varphi_2}{dt^{r-1}}(t_0) \right)$$
$$= \Psi_{t_0}(\varphi_1) + \Psi_{t_0}(\varphi_2).$$

Similarly,

$$\Psi_{t_0}(\alpha\varphi) = \left((\alpha\varphi)(t_0),\ \frac{d(\alpha\varphi)}{dt}(t_0), \dots, \frac{d^{r-1}(\alpha\varphi)}{dt^{r-1}}(t_0) \right)$$

$$= \left(\alpha\varphi(t_0),\ \alpha\frac{d\varphi}{dt}(t_0), \dots, \alpha\frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) \right)$$

$$= \alpha\,\Psi_{t_0}(\varphi).$$

Ψ_{t_0} is surjective (onto) by the Existence part of the Existence and Uniqueness Theorem:

For any vector $(\varphi_0^{(0)}, \varphi_0^{(1)}, \dots, \varphi_0^{(r-1)}) \in \mathbb{R}^r$, there exists a solution $\varphi \in Sol(H)$ satisfying the initial conditions

$$\varphi(t_0) = \varphi_0^{(0)}, \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = \varphi_0^{(r-1)}$$

and

$$\Psi_{t_0}(\varphi) = \left(\varphi(t_0), \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) \right)$$

$$= \left(\varphi_0^{(0)}, \dots, \varphi_0^{(r-1)} \right).$$

Ψ is injective (one-to-one) by the Uniqueness part of the Existence and Uniqueness Theorem:

If $\Psi_{t_0}(\varphi_1) = \Psi_{t_0}(\varphi_2)$, then

$$\left(\varphi_1(t_0), \frac{d\varphi_1}{dt}(t_0), \dots, \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) \right)$$
$$= \left(\varphi_2(t_0), \frac{d\varphi_2}{dt}(t_0), \dots, \frac{d^{r-1}\varphi_2}{dt^{r-1}}(t_0) \right)$$

So φ_1 and φ_2 are solutions in $Sol(H)$ satisfying the same initial conditions. By Uniqueness, we conclude that $\varphi_1 = \varphi_2$.

Alternative argument for injectivity:

It's enough to check that $\ker \Psi_{t_0} = \{0\}$.

$$\ker \Psi_{t_0} = \{ \varphi \in Sol(H) : \Psi_{t_0}(\varphi) = (0, \dots, 0) \}$$
$$= \{ \varphi \in Sol(H) : \varphi(t_0) = 0, \frac{d\varphi}{dt}(t_0) = 0, \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = 0 \}$$

Now, the function $\varphi(t) = 0$ for all $t \in I$ satisfies the initial conditions

$$\varphi(t_0) = 0, \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = 0,$$

and by Uniqueness is the only such function. Therefore,

$$\ker \Psi_{t_0} = \{0\}$$

zero function.

Ψ_{t_0} is a linear map that is bijective (both injective and surjective), so it is an isomorphism.

Isomorphic vector spaces are "identical"; in particular, two isomorphic vector spaces have the same dimension.

Therefore, we have shown that the space of solutions of

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)\,y = 0$$

over I has dimension r

(under the assumption that $a_{r-1}(t), \ldots, a_0(t)$ are continuous functions of t over I).

$\square$