MThe 237
Lecture 07
Sept. 26, 2017

Topics: • Proof of Criterion of Exactness
• Implicit Function Theorem

Last time, we defined exact equations:

Def. A differential equation of the form

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0,$$

for some function $G:\mathbb{R}^2 \to \mathbb{R}$, is called **exact**.

and stated the following criterion:

Theorem. Suppose the functions $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}(x,y)$ and $\frac{\partial N}{\partial x}(x,y)$ exist and are continuous in an open rectangle R. If

$$\boxed{\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)} \quad \text{for all } (x,y) \in R$$

then there exists a function $G:\mathbb{R}^2 \to \mathbb{R}$, with

$$\frac{\partial G}{\partial x} = M(x,y), \quad \frac{\partial G}{\partial y} = N(x,y).$$

Corollary. If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous in R, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \text{The differential equation } M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ is exact in } R$$

Here is a proof of the theorem.

Proof. We need to find a function $G : R \to \mathbb{R}$ such that

$$\frac{\partial G}{\partial x} = M, \quad \frac{\partial G}{\partial y} = N \quad \text{in } R.$$

For some motivation, imagine we have a function G as above. Then, for any fixed y, we have

$$G(x,y) - G(x_0,y) = \int_{x_0}^{x} M(t,y)\,dt$$

by the fundamental theorem of calculus. Adding $G(x_0,y)$ to both sides,

$$G(x,y) = \int_{x_0}^{x} M(t,y)\,dt + G(x_0,y).$$

This motivates looking for G of the form

$$G(x,y) = \int_{x_0}^{x} M(t,y)\,dt + h(y) \qquad (*)$$

for some function h.

With G of this form, we have

$$\frac{\partial G}{\partial x} = M(x,y) + 0$$

for any h.

The other condition that needs to be satisfied is

$$\frac{\partial G}{\partial y} = N(x,y).$$

Taking the partial with respect to y of $(*)$, we find

$$\frac{\partial G}{\partial y} = \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)\,dt + h'(y)$$

If this is equal to $N(x,y)$, we obtain

$$h'(y) = N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)\,dt.$$

If the right side is a function of y only, we can integrate with respect to y and solve h.

To check if the right side is a function of y only, we compute

$$\frac{\partial}{\partial x}\left(N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)\,dt \right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)\,dt$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} \int_{x_0}^{x} M(t,y)\,dt$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} M(x,y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 \quad \text{by hypothesis}$$

Therefore, $N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)\,dt$ is a function of y only.

$$h(y) = \int_{y_0}^{y} \left(N(x,s) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,s)\,dt \right) ds$$

The function

$$G(x,y) = \int_{x_0}^{x} M(t,y)\,dt + \int_{y_0}^{y} \left(N(x,s) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,s)\,dt \right) ds$$

satisfies $\frac{\partial G}{\partial x} = M$, $\frac{\partial G}{\partial y} = N$. ∎

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Separable Equations are Exact

We can think of exact equations as a broadening of the category of separable equations.

- Separable equations have solutions given by level curves $\int m(x)\,dx - \int n(y)\,dy = C$; exact equations are exactly the differential equations whose solutions are level curves.

- We can check that separable equations satisfy the exactness criterion:

$$n(y) \frac{dy}{dx} = m(x)$$

can be rewritten as

$$\underbrace{m(x)}_{M(x,y)} - \underbrace{n(y)}_{N(x,y)} \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} m(x) = 0 \; ; \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-n(y)) = 0.$$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The usual approach to solving exact equations reduces to the separation of variables in the separable case.

$$m(x) - n(y)\frac{dy}{dx} = 0$$

1. Integrate $M(x,y)$ with respect to x, obtaining

$$\int m(x)\,dx + h(y)$$

2. Take partial of the result with respect to y

$$0 + h'(y)$$

Match with $N(x,y) = -n(y)$.

We need $h'(y) = -n(y)$. Integrating with respect to y,

$$h(y) = \int -n(y)\,dy$$

3. Therefore, solutions are given by

$$C = \int m(x)\,dx + h(y)$$

$$= \int m(x)\,dx + \int -n(y)\,dy$$

or $$\int m(x)\,dx = \int n(y)\,dy + C$$

This is what we would have obtained by separation of variables.

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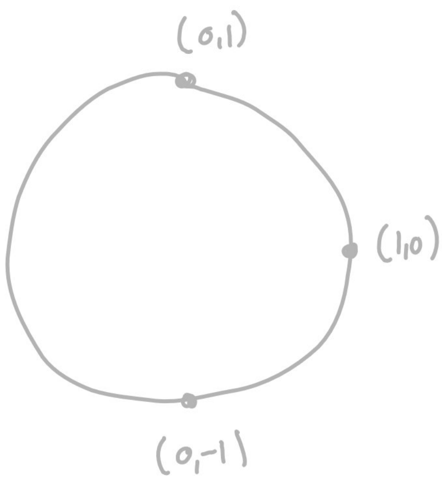
Implicit Function Theorem

Since we are allowing solutions in implicit form, we should understand when, at least in principle, we can solve $G(x,y)=C$ for y as a function of x in a neighbourhood of some point.

Let's begin by looking at two examples.

1. $$\underbrace{x^2+y^2}_{G(x,y)}=1 \quad \text{(a circle)}$$



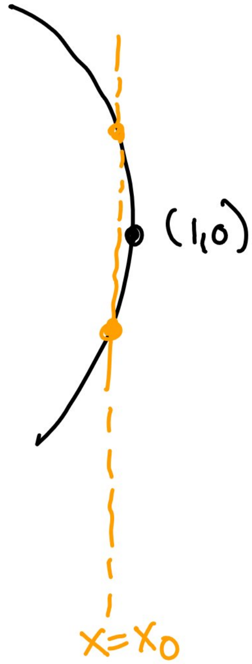
Near $(0,1)$, $G(x,y)=1$ can be described by the graph of the function

$$y(x)=\sqrt{1-x^2}$$

Near $(0,-1)$, $G(x,y)=1$ can be described by the graph of the function

$$y(x)=-\sqrt{1-x^2}.$$

Near $(1,0)$, we can't describe $G(x,y)=1$ as the graph of a function $y(x)$

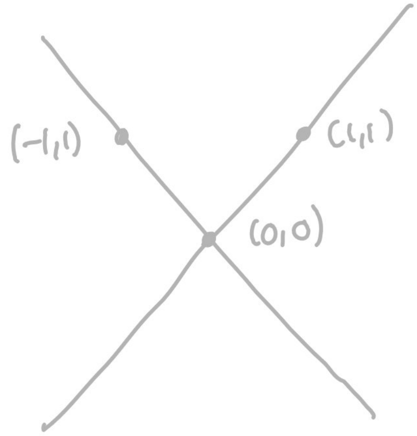


However close we stay to (1,0) on $G(x,y)=1$, a vertical line intersects $G(x,y)=1$ in two points, hence can't be the graph of a function $y(x)$.

What is $y(x_0)$? To describe $G(x,y)=1$, need $y(x_0)$ to take two values, which is not possible for a function.

We learn from this example that one situation when we can't describe $G(x,y)=C$ by a graph $y=y(x)$ is when $G(x,y)=C$ has a <u>vertical tangent line</u>.

2. $\overbrace{x^2-y^2}^{G(x,y)}=0$



Near $(1,1)$, $G(x,y)=0$ can be described by the graph of $y(x)=x$.

Near $(-1,1)$, $G(x,y)=0$ can be described by the graph of $y(x)=-x$.

Near $(0,0)$, $G(x,y)=0$ can't be described by the graph of a function.

We learn from this example that one situation when we can't describe $G(x,y)=c$ by a graph $y=y(x)$ is when $G(x,y)=c$ has a <u>self-intersection</u> or another type of <u>singularity</u>.

Vertical tangent line: $\frac{\partial G}{\partial x}(x_0,y_0)\neq 0$, $\frac{\partial G}{\partial y}(x_0,y_0)=0$
at (x_0,y_0)

Singularity at (x_0,y_0): $\frac{\partial G}{\partial x}(x_0,y_0)=0$, $\frac{\partial G}{\partial y}(x_0,y_0)=0$.

We see that both situations can't happen if $\frac{\partial G}{\partial y}(x_0,y_0)\neq 0$.

The following theorem can be interpreted as saying that these are the only situations when $G(x,y)=c$ can't be described by the graph $y=y(x)$:

Theorem [Implicit Function Theorem].

Let (x_0, y_0) be a point on the curve $G(x,y) = C$ (so that $G(x_0,y_0) = C$). If $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, then there exists a nonempty open interval $I \subset \mathbb{R}$ containing x_0 and a function $y : I \to \mathbb{R}$ with $y(x_0) = y_0$ and

$$G(x, y(x)) = 0 \quad \text{for all } x \in I.$$

If $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ exist and are continuous, then $y : I \to \mathbb{R}$ is differentiable

Example: For the circle $x^2 + y^2 = 1$, $G(x,y) = x^2 + y^2$,

$$\frac{\partial G}{\partial y} = 2y$$

Implicit function theorem says we can describe $x^2+y^2=1$ by the graph of a function $y = y(x)$ except possibly around points where

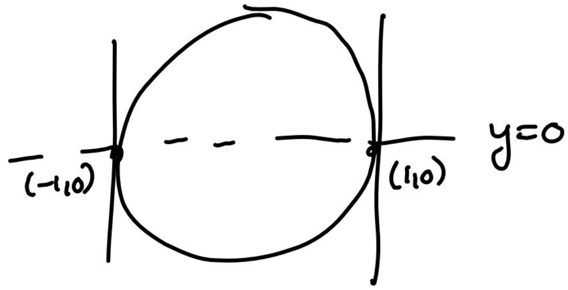
$$\frac{\partial G}{\partial y} = 0.$$

$\frac{\partial G}{\partial y} = 2y$ is equal to 0 if and only if $y = 0$.

The only two points on $x^2+y^2=1$ with $y=0$ are

$(1,0)$ and $(-1,0)$. At these points,

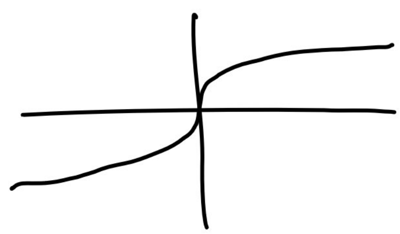
$x^2+y^2=1$ indeed has a vertical tangent.



Example. The implicit function theorem gives a one-way implication. Here is an example to show this.

Let $G(x,y) = y^3 - x$.

Then $G(x,y) = 0$ looks like



$$\frac{\partial G}{\partial y} = 3y^2, \text{ so } \frac{\partial G}{\partial y}(0,0) = 0.$$

Indeed, the curve $G(x,y)=0$ has a vertical tangent at $(0,0)$. However, the curve $G(x,y)=0$ coincides with the graph of the function

$y(x) = \sqrt[3]{x}$, defined for all $\mathbb{R}$.

In this example, the vertical tangent line does not preclude finding a function whose graph describes the level curve. Roughly, the reason is that the curve does not "stay on the same side" of the vertical tangent.