Mthe 237
Lecture 06
Sept. 22, 2017

Topics: Implicit Solutions
Exact Differential Equations

Example. $$y\frac{dy}{dx} = -x$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$y^2 = -x^2 + C$$

Up to now, our algorithm for solving separable equations called for solving for y as a function of x.

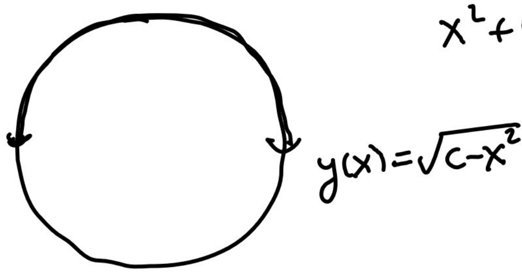
However, by adding x^2 to both sides, we obtain $x^2+y^2=C$, which describes both solutions

$$y(x) = \pm\sqrt{C-x^2}$$ at once.

A different issue is that it may be practically impossible to solve a relation of the form $G(x,y)=C$ for y as a function of x.

For these reasons, we would like to allow descriptions of solutions of the form $G(x,y)=C$, that "contain" graphs of valid solutions.

In the example,



$x^2+y^2=c$ describes a circle of radius $\sqrt{c}$.

It contains the graph of the solution $y(x)=\sqrt{c-x^2}$ as the upper semicircle.

<u>Def.</u> An equation $G(x,y)=c$ is called an <u>implicit solution</u> to the differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^r y}{dx^r}\right)=0 \qquad (*)$$

over an open interval $I \subset \mathbb{R}$ if there exists a solution $\varphi: I \to \mathbb{R}$ of $(*)$ that also satisfies

$$G(x, \varphi(x)) = c \quad \text{for all } x \in I.$$

In the example, $\underbrace{x^2+y^2}_{G(x,y)}=c$

is an implicit solution of $y\frac{dy}{dx}=-x$

over $(-1,1)$.

<u>Note</u>:
In the terminology of APSC 172, $G(x,y)=c$ is a <u>level curve</u> of the function $G: \mathbb{R}^2 \to \mathbb{R}$.

Question: Is every level curve $G(x,y) = c$ an implicit solution of some differential equation?

Answer: Yes, it is a solution of

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0 \,!$$

Two arguments:

1. Suppose the graph of $y(x)$ lies on $G(x,y) = c$.

   Then $G(x, y(x)) = c$.

   By implicit differentiation,

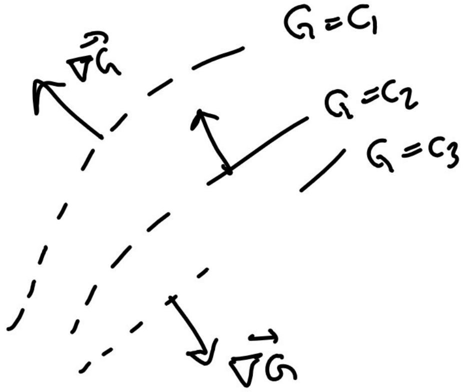
$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0.$$

2. Suppose the graph of $y(x)$ lies on $G(x,y) = c$, and is parametrized by $t \mapsto (x(t), y(t))$.

Reminder: The gradient $\vec{\nabla} G$ of G is the vector-valued function

$$\vec{\nabla} G(x,y) = \left(\frac{\partial G}{\partial x}(x,y), \frac{\partial G}{\partial y}(x,y)\right).$$

An important property of the gradient is that it is always perpendicular to the level curves of G.



Intuitively, the reason for this is that $\vec{\nabla} G$ is the direction of maximal change of G, while G is constant along level curves. So moving any amount along a level curve would not increase G.

Since $\vec{\nabla} G$ is perpendicular to a level curve, and $(\dot{x}(t), \dot{y}(t))$ is parallel (tangent), we have

$$0 = \vec{\nabla} G \cdot (\dot{x}(t), \dot{y}(t)) = \frac{\partial G}{\partial x} \dot{x}(t) + \frac{\partial G}{\partial y} \dot{y}(t).$$

Dividing by $\dot{x}(t)$,

$$0 = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx}.$$

(Here we are applying the fact $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$.)

<u>Def.</u> A differential equation of the form
$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx} = 0$$
for some $G : \mathbb{R}^2 \to \mathbb{R}$ is called <u>exact</u>.

This is the equation satisfied by all level curves $G(x,y) = c$.

Conversely, if $y(x)$ is a solution to an exact equation,

$$0 = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx} = \frac{d}{dx}\left(G(x, y(x)) \right), \quad \text{so} \quad G(x, y(x)) = c.$$

Q: How do we recognize when a diff. eq. of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact?

Reminder: Second partial derivatives are denoted

$$\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right), \qquad \frac{\partial^2 G}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right),$$

$$\left(\text{and} \quad \frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x}\right), \qquad \frac{\partial^2 G}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial y}\right)\right).$$

Theorem If $\frac{\partial^2 G}{\partial x \partial y}$ and $\frac{\partial^2 G}{\partial y \partial x}$ are both continuous in an open rectangle R, then

$$\frac{\partial^2 G}{\partial x \partial y}(x,y) = \frac{\partial^2 G}{\partial y \partial x}(x,y) \quad \text{for all } (x,y) \in R.$$

Example: $f(x,y) = \cos(x)\,y^2 + 3x^2y^2$

$$\frac{\partial f}{\partial x} = -\sin(x)\,y^2 + 6xy^2, \qquad \frac{\partial f}{\partial y} = 2\cos(x)\,y + 6x^2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2\sin(x)\,y + 12xy, \qquad \frac{\partial^2 f}{\partial x \partial y} = -2\sin(x)\,y + 12xy.$$

Equal !!

If $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact, so that

$$M(x,y) = \frac{\partial G}{\partial x}(x,y)$$
$$N(x,y) = \frac{\partial G}{\partial y}(x,y)$$

for some $G : \mathbb{R}^2 \to \mathbb{R}$, and $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in some open rectangle, then by the theorem above

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right) = \frac{\partial^2 G}{\partial y \partial x}$$
$$= \frac{\partial^2 G}{\partial x \partial y}$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is a <u>necessary</u> condition for the equation $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ to be exact.

This provides a test for exactness:
if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, we know eq. is <u>not</u> exact.

The converse is also true:

<u>Theorem</u> If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are all continuous on an open rectangle R, and

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y) \quad \text{for all } (x,y) \in R,$$

then there exists $G : R \to \mathbb{R}$ with

$$\frac{\partial G}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial G}{\partial y}(x,y) = N(x,y)$$

for all $(x,y) \in R$.

<u>How to solve exact equations</u>:

Solutions will be described implicitly by
$G(x,y) = c$, for some c

so the task is to find G.

<u>Method 1</u>. Guess what G is. (Often possible)
Look for G with $\frac{\partial G}{\partial x} = M$, $\frac{\partial G}{\partial y} = N$.

<u>Method 2</u>. (In an example)

$$\underbrace{(y\cos(x) + 2xe^y)}_{M(x,y)} + \underbrace{(\sin(x) + x^2e^y - 1)}_{N(x,y)}\frac{dy}{dx} = 0.$$

<u>Test</u>: $\frac{\partial M}{\partial y} = \cos(x) + 2xe^y$ $\quad \frac{\partial N}{\partial x} = \cos(x) + 2xe^y$. Equal $\Rightarrow$ Exact.

1. Integrate $M(x,y)$ with respect to x.

(Looking for G such that $\frac{\partial G}{\partial x} = M$, so this is reasonable to try)

Obtain

$$y\sin(x) + x^2e^y + \underbrace{h(y)}_{} \qquad (**)$$

"A family of constants of integration, one for each value of y"

2. Take partial derivative with respect to y.

$$\sin(x) + x^2 e^y + h'(y)$$

3. Expect this to equal $\frac{\partial G}{\partial y} = N$, so match terms with $N(x,y)$ to obtain an equation for $h'(y)$.

$$N(x,y) = \sin(x) + x^2 e^y - 1 \text{ , so}$$

$h'(y) = -1$, or $h(y) = -y + C$. Plug back into (**)

The solutions are given implicitly by

$$\underbrace{y \sin(x) + x^2 e^y - y}_{G(x,y)} = c$$

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1. Alternatively, we could have started with integrating $N(x,y)$ with respect to y:

$$\sin(x)\, y + x^2 e^y - y + h(x)$$

2. $\frac{\partial}{\partial x}$ of the result

$$\cos(x)\, y + 2xe^y + h'(x)$$

3. Match with $M(x,y)$.

$$M(x,y) = \cos(x)\, y + 2xe^y$$

This implies $h'(x) = 0$, so $h(x) = C$.

$G(x,y) = y\sin(x) + x^2 e^y - y$, as before.

<u>Example:</u> For a function of the form

$$G(x,y) = \int n(y)\,dy - \int m(x)\,dx,$$

$$\frac{\partial G}{\partial x} = -m(x)$$

$$\frac{\partial G}{\partial y} = n(y)$$

The level curves $G(x,y) = c$ are implicit solutions of the differential equation

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0 \quad \text{or}$$

$$-m(x) + n(y)\frac{dy}{dx} = 0 \quad \text{or}$$

$$n(y)\frac{dy}{dx} = m(x).$$

This is why separation of variables works!