MThe 237
Lecture 05
Sept. 20, 2017

Topics: Change of Variables
Homogeneous DEs

Sometimes, a change of variable (also referred to as substitution) may transform a differential equation into one that is simpler to solve.

<u>Example.</u> $\frac{dy}{dx} = x+y, \qquad y(0)=0.$

This is one of the simplest examples of a non-separable equation (as proved in the optional problem on HW 2).

Let's define a new function v by

$$v(x) = x + y(x)$$

This function is yet unknown, like $y(x)$, but from the differential equation for $y(x)$ we can see that $v(x)$ also satisfies a differential equation:

We have $y(x) = v(x) - x$, so that

$\frac{dy}{dx}(x) = \frac{dv}{dx}(x) - 1$ by implicit differentiation,

so the diff. eq.

$\frac{dy}{dx} = x+y$ becomes $\left(\frac{dv}{dx} - 1\right) = v$

or $dv/dx = v+1$. Separable!

Separating variables,

$$\frac{1}{v+1}\,\frac{dv}{dx} = 1$$

Integrating both sides,

$$\ln|v+1| = x + C$$

$$v+1 = Ce^{x}$$

$$v = Ce^{x} - 1$$

The initial condition on y induces an initial condition on v: $\quad v(0) = 0 + y(0) = 0.$

Hence, $\quad 0 = v(0) = Ce^{0} - 1 \implies C = 1.$

So

$$x + y = v = e^{x} - 1, \quad \text{and}$$

$$y(x) = e^{x} - 1 - x.$$

(We did not have to find the induced initial condition on v. The constant C could have been determined after changing coordinates back to x and y.)

We can check that this y solves the diff. eq.

$$\frac{dy}{dx} = e^{x} - 1 = (e^{x} - 1 - x) + x$$

$$= y + x.$$

(The check is not a part of the solution.)

In general, look for a function

$$v = h(x, y(x))$$

The technique works well when h is 1-1 (hence invertible) and there exists a g so that

$$y = g(x, v(x))$$

(We can think of finding an inverse as solving for y in terms of x and v.)

By implicit differentiation,

$$\frac{dy}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial v}\frac{dv}{dx}$$

(Note: this is a special case of the multivariable chain rule)

If $y(x_0) = y_0$ is the initial condition on y, there is an induced initial condition on v:

$$v(x_0) = h(x_0, y(x_0)) = h(x_0, y_0).$$

(In the example, $h(x, y(x)) = x + y(x)$,

$$g(x, v(x)) = v(x) - x$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial v}\frac{dv}{dx} = (-1) + (1)\frac{dv}{dx}$$

$$= \frac{dv}{dx} - 1.$$
)

There are few general rules for when a substitution will work well, finding a good one is a matter of trying things and expreriventing.

There are, however, a few classes of differential equations with standard substitutions. One of these classes is

Homogeneous equations

Def. A function $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$$(x,y) \longmapsto F(x,y)$$

is called homogeneous of degree d if for all $c>0$,

$$F(cx, cy) = c^d F(x,y).$$

Examples

- $F(x,y) = x^2+y^2$.

$$F(cx, cy) = (cx)^2+(cy)^2 = c^2(x^2+y^2)$$
$$= c^2F(x,y)$$

Homogeneous of degree 2.

- $F(x,y) = x+y+\sqrt{x^2+y^2}$

$$F(cx,cy) = (cx)+(cy)+\sqrt{(cx)^2+(cy)^2}$$
$$= c\left(x+y+\sqrt{x^2+y^2}\right)$$
$$= c\,F(x,y)$$

Homog. of degree 1

- $F(x,y) = e^{y/x}$. $\quad F(cx,cy) = e^{cy/cx} = e^{y/x} = F(x,y)$

Homogeneous of degree 0.

- $F(x,y) = (x^3 + 2x^2 y)^{1/5}$. Homogeneous of degree $3/5$

Non homogeneous: $F(x,y) = x^2 + x$.

$$F(cx,cy) \neq c^d F(x,y) \text{ for any } d.$$

A polynomial in x and y is homogeneous if and only if all terms have the same degree.

<u>Def</u>. A differential equation

$$M(x,y)\frac{dy}{dx} = N(x,y)$$

is called <u>homogeneous</u> if M, N are both homogeneous, <u>of the same</u> degree.

Suppose M, N are homogeneous of equal degree d

Then $$\frac{dy}{dx} = \frac{N(x,y)}{M(x,y)} = \frac{x^d N(1,\frac{y}{x})}{x^d M(1,\frac{y}{x})} = \underbrace{\frac{N(1,\frac{y}{x})}{M(1,\frac{y}{x})}}_{\text{Function of } \frac{y}{x} \text{ only.}}$$

This suggests changing variables to $v = \frac{y}{x}$.

$y = xv$, $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ ← Product rule

Prop. The change of variables $v = \frac{y}{x}$ makes a homogeneous equation separable.

Computation: By above,

$$\frac{dy}{dx} = \frac{N(1, \frac{y}{x})}{M(1, \frac{y}{x})}, \quad \text{or}$$

Rewriting in terms of x and v,

$$v + x\frac{dv}{dx} = \frac{N(1,v)}{M(1,v)}$$

$$x\frac{dv}{dx} = \frac{N(1,v)}{M(1,v)} - v = \frac{N(1,v) - v\,M(1,v)}{M(1,v)}$$

This is separable:

$$\frac{M(1,v)}{N(1,v) - v\,M(1,v)} \frac{dv}{dx} = \frac{1}{x}.$$

(This complicated expression is not worth remembering. The key is the substitution $v = \frac{y}{x}$)

Let's now solve the drone equation:

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2+y^2}}{x}, \quad x>0, \quad \underbrace{y(a)=0.}_{\text{Drone passes the point } (a,0)}$$

This is homogeneous!

(Both $M(x,y) = x$ and $N(x,y) = y - k\sqrt{x^2+y^2}$ are homogeneous of degree 1).

$$v = \frac{y}{x}; \quad y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$

$$\boxed{v(a) = \frac{0}{a} = 0}$$

$$v + x\frac{dv}{dx} = \frac{vx - k\sqrt{x^2+(vx)^2}}{x}$$

$$= v - k\sqrt{1+v^2}$$

So $$\frac{1}{\sqrt{1+v^2}}\frac{dv}{dx} = -\frac{k}{x}.$$

Computing $\int \frac{dv}{\sqrt{1+v^2}}$: Let $v = \sinh(t)$

$dv = \cosh(t)\,dt$

Since $\cosh^2(t) - \sinh^2(t) = 1$,

$\cosh^2(t) = 1 + \sinh^2(t)$.

Therefore,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\cosh(t)\,dt}{\sqrt{1+\sinh^2(t)}} = \int \frac{\cosh(t)}{\cosh(t)}\,dt = t$$

So separation of variables gives

$$t = -k\ln(x) + c.$$

Applying sinh to both sides,

$$v = \sinh(t) = \sinh(-k\ln(x) + c).$$

Reminder: $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

Notice that

$$\sinh(x) = 0 \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = -x \Rightarrow x = 0 \qquad (*)$$

So, the initial condition $v(a) = 0$ implies that

$0 = \sinh(-k\ln(a) + c)$, so

$$c = k\ln(a) \text{ by } (*).$$

Since
$$-k\ln(x) + k\ln(a) = -k\ln\left(\frac{x}{a}\right) = \ln\left(\left(\frac{x}{a}\right)^{-k}\right),$$
we have

$$v = \sinh\left(\ln\left(\frac{x}{a}\right)^{-k}\right) = \frac{\exp\left(\ln\left(\frac{x}{a}\right)^{-k}\right) - \exp\left(-\ln\left(\frac{x}{a}\right)^{-k}\right)}{2}$$

$$= \frac{1}{2}\left(\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right)$$

Finally, $v = \frac{y}{x}$, so

$$y(x) = x \cdot \frac{1}{2}\left(\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right)$$

$$= \frac{1}{2a}\left(\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right).$$

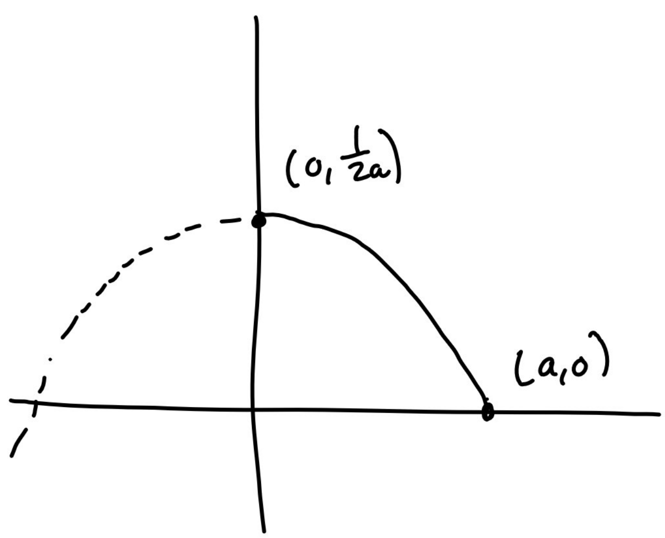
When we derived the drone equation, we put

$$k = \frac{v}{w}.$$

When $k=1$, that is $v=w$, the above function becomes

$$\frac{1}{2a}\left(1 - \left(\frac{x}{a}\right)^2\right).$$

This is a parabola (down-facing)



when $v=w$, the drone drifts past the origin, as we may have expected.