Mthe 237
Lecture 04
Sept. 19, 2017

Topic: Existence and Uniqueness for First-Order ODE.

Let's look at the example

$$x\frac{dy}{dx} = 3y,$$

and ask what can happen if we ignore the restrictions on the domain that arise in separation of variables.

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x}$$

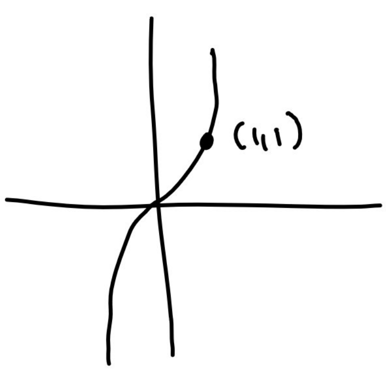
$$\int \frac{dy}{y} = \int \frac{3}{x}\,dx$$

$$\ln|y| = 3\ln|x| + C$$

$$y = Cx^3,$$ C arbitrary constant (could be 0 or negative)

Consider three initial conditions:

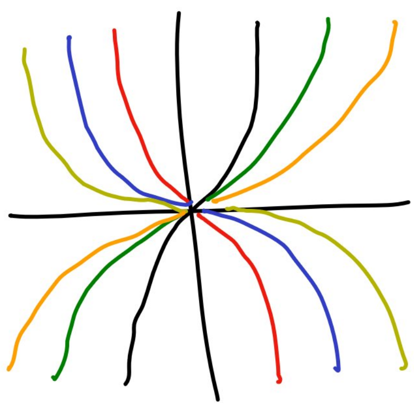
① $y(1)=1$. $1 = y(1) = C\cdot 1^3$, so $C=1$.



There is only one solution in the family found by separation of variables that satisfies the initial condition.

[The solution exists and is unique]

(2) $y(0) = 0.$ $0 = y(1) = C \cdot 0^3.$ No condition on C!



Every solution in the family found by separation of variables satisfies the initial condition.

[A solution exists, but is not unique]

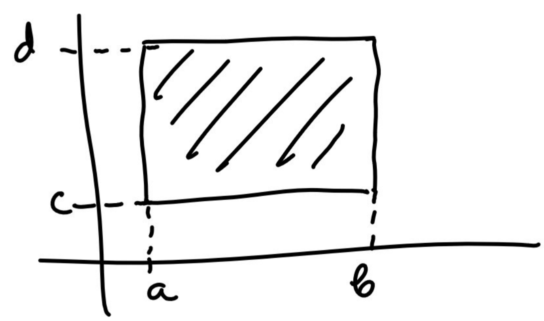
(3) $y(0) = 1.$ $1 = y(0) = C \cdot 0^3.$ Impossible.

NO solution in the family satisfies the initial condition.

[A solution does not exist]

We would like to find general criteria that rule out this type of behaviour of solutions.

Closed rectangle in $\mathbb{R}^2$: $[a,b] \times [c,d] = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} a \le x \le b, \\ c \le y \le d \end{array} \right\}$



Open rectangle: $(a,b)\times(c,d) = \left\{ (x,y)\in\mathbb{R}^2 : \begin{array}{l} a<x<b \\ c<y<d \end{array} \right\}$.

The <u>interior</u> of a closed rectangle $[a,b]\times[c,d]$ is the open rectangle $(a,b)\times(c,d)$.

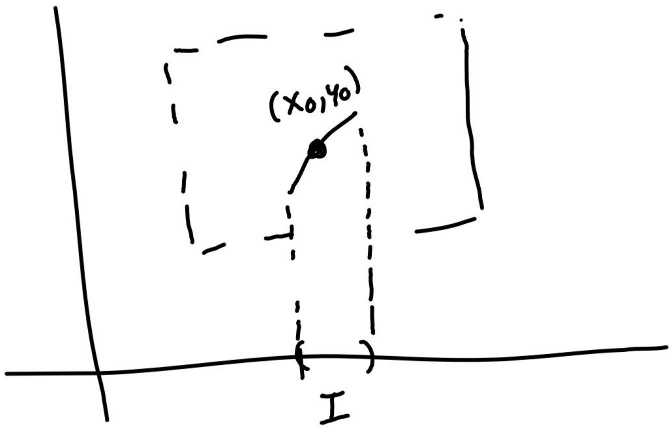
<u>Theorem</u> [Uniqueness and Existence for First-Order Differential Equations]

Consider the differential equation

$$(*) \qquad \frac{dy}{dx} = F(x,y), \qquad y(x_0) = y_0$$

<u>Existence</u>: Suppose that F is continuous on a closed rectangle $[a,b]\times[c,d]$ that contains (x_0,y_0) in its interior. Then there exists a nonempty open interval $I\subset[a,b]$, and a solution $\varphi: I\to\mathbb{R}$ of $(*)$ over I.

<u>Uniqueness</u>: If in addition $\frac{\partial F}{\partial y}$ is continuous on $[a,b]\times[c,d]$ that contains (x_0,y_0) in its interior, then there exists a <u>unique</u> solution of $(*)$ over some nonempty $I\subset[a,b]$.

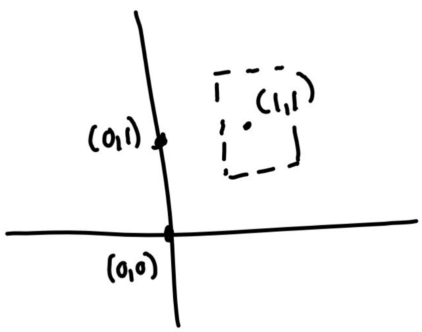


In our example,

$$F(x,y) = \frac{3y}{x}$$

has a non-removable discontinuity over the y-axis.

Therefore, we <u>can</u> find a rectangle as in the statement of the Theorem about (1,1), but not about (0,1) and (0,0).



This consistent with what we found: there was a unique solution through (1,1), but many through (0,0) and none through (0,1).

<u>Remark</u>. The theorem is a <u>one-way</u> implication. If the hypotheses do not hold, we do not know whether the conclusions hold without further investigation.

Another example: $\frac{dy}{dx} = y^2, \quad y(0)=1.$

$$\frac{1}{y^2}\frac{dy}{dx} = 1$$

$$-\frac{1}{y} = x + C, \quad y = \frac{1}{1-x}.$$

Even though both
$F(x,y) = y^2$ and $\frac{\partial F}{\partial y} = 2y$
are continuous for all $(x,y) \in \mathbb{R}^2$, the solution does not exist for all x.

When can we ignore the restrictions on the domain that come up in separation of variables?

$$\frac{dy}{dx} = m(x)u(y)$$

Suppose that $u(y_0)=0$.

Then the constant solution $y(x)=y_0$ may not be an element of the family produced by separation of variables. These solutions should be checked separately.

If we are in a region where $F = m(x)u(y)$ and $\frac{\partial F}{\partial y} = m(x)u'(y)$ are continuous, then other solutions cannot cross the constant solutions, and we do not need to worry further about restrictions on domain they impose.

[More generally, can conclude this if know solutions are unique]

If we are not in a region where uniqueness is guaranteed, ignoring restrictions on the domain can lead to unexpected behaviour:

$$\frac{dy}{dx} = \frac{3y}{x}, \qquad y(0)=0.$$

There is a 2-parameter family of solutions (even though the equation is first order)

$$y(x) = \begin{cases} c_1 x^3, & x \geq 0 \\ c_2 x^3, & x < 0. \end{cases}$$

A single initial condition is not sufficient to pick out a solution.

So, blindly applying separation of variables may lead to unexpected behaviour of solutions, such as lost solutions.

Key idea in proof of the Existence and Uniqueness Theorem:

Reformulate the differential equation

(1) $\frac{dy}{dx} = F(x,y)$, $y(x_0)=y_0$

as an integral equation

(2) $y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$.

A function satisfies (1) iff it satisfies (2):

(1) ⇒ (2):

$$y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$$

$$= y_0 + \int_{x_0}^{x} \frac{dy}{dt}(t)\, dt \underset{\substack{\uparrow \\ \text{Fundamental} \\ \text{Theorem of Calculus}}}{=} y_0 + (y(x) - y(x_0)) = y(x).$$

(2) ⇒ (1): Differentiate

$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$ to get

$$\frac{dy}{dx} = 0 + F(x, y(x)).$$

Construct a solution iteratively:

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0(t))\, dt$$

$$\vdots$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(t, y_n(t))\, dt.$$

$$\vdots$$

<u>Claim</u>: The sequence of functions $y_n(x)$ "converges" to a solution to the diff. eq.

$$\frac{dy}{dx} = F(x, y), \qquad y(x_0) = y_0$$

<u>Example</u>. $\frac{dy}{dx} = y$, $y(0) = 1$. Expect: $y(x) = e^x$.

$$y_0(x) = y_0 = 1,$$

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0(t))\, dt$$

$$= 1 + \int_0^x 1\, dt = 1 + x,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} F(t, y_1(t))\, dt$$

$$= 1 + \int_0^x 1 + x\, dx$$

$$= 1 + x + \frac{x^2}{2}$$

Proceeding in this way,

$$y_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

As $n \to \infty$, $y_n(x)$ converges to

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$