Mthe 237
Lecture 03
Sept. 15, 2017

For a revised discussion of the example

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x}, \qquad x \neq 0,\ y \neq 0$$

in light of the uniqueness and existence theorem for ODEs, please see notes of lecture 04.

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A basic fact about solutions to differential equations (of order > 0) is that, being differentiable, they are necessarily continuous.

This requirement, together with the requirement that solutions are real-valued, sometimes places restrictions on the domain of definition of solutions, beyond the restrictions coming from the domain of the differential equation.

<u>Example.</u> $$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = 1.$$

The equation is separable. Separating variables,

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}.$$

Integrating both sides,

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}}\,dx \longleftarrow$$

Let $u = 1+x^2$
$du = 2x\,dx$

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{2}\frac{du}{\sqrt{u}}$$
$$= \sqrt{u} + C$$
$$= \sqrt{1+x^2} + C$$

$$-\frac{1}{2}\frac{1}{y^2} = \sqrt{1+x^2} + C$$

Applying the initial condition,

$$-\frac{1}{2}\frac{1}{1^2} = \sqrt{1+0^2} + C = 1 + C,$$

$$\text{so} \quad C = -3/2.$$

Solving for y as a function of x,

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}},$$

$$y = \pm\sqrt{\frac{1}{3-2\sqrt{1+x^2}}}.$$

The branch of the solution should satisfy

$$y(0) = 1.$$

$$\pm\sqrt{\frac{1}{3-2\sqrt{1+0^2}}} = \pm\sqrt{\frac{1}{3-2}} = \pm 1.$$

Therefore, the branch satisfying $y(0) = 1$ is

$$y(x) = \sqrt{\frac{1}{3-2\sqrt{1+x^2}}}.$$

This is well-defined when:

i) $3 - 2\sqrt{1+x^2} \neq 0$ (can't divide by 0)

ii) $3 - 2\sqrt{1+x^2} \geq 0$ (looking for real-valued solution)

In this case, the two conditions can be combined to

$$3 - 2\sqrt{1+x^2} > 0.$$

This is equivalent to

$$2\sqrt{1+x^2} < 3$$

$$\Leftrightarrow \quad \sqrt{1+x^2} < \frac{3}{2}$$

$$\Rightarrow \quad 1 + x^2 < \frac{9}{4}$$

$$\Leftrightarrow \quad x^2 < 5/4, \quad \text{so} \quad -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

This is the maximal domain over which the solution is defined.

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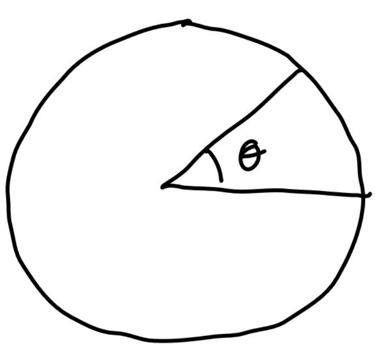
For the rest of the lecture, we took some time to solve the catenary equation

$$\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

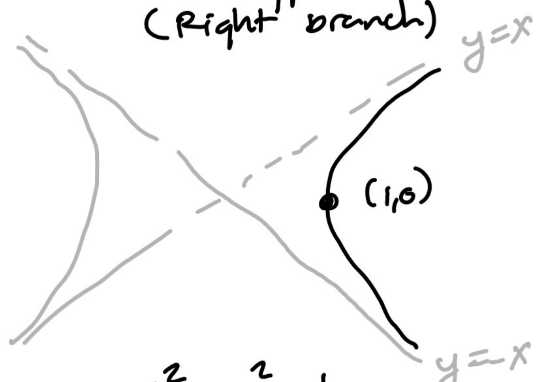
As a prelude, we need to define a pair of functions that are analogous (in a precise sense) to the usual trigonometric functions cos and sin.

Unit circle



$x^2+y^2=1$

Unit hyperbola (Right branch)

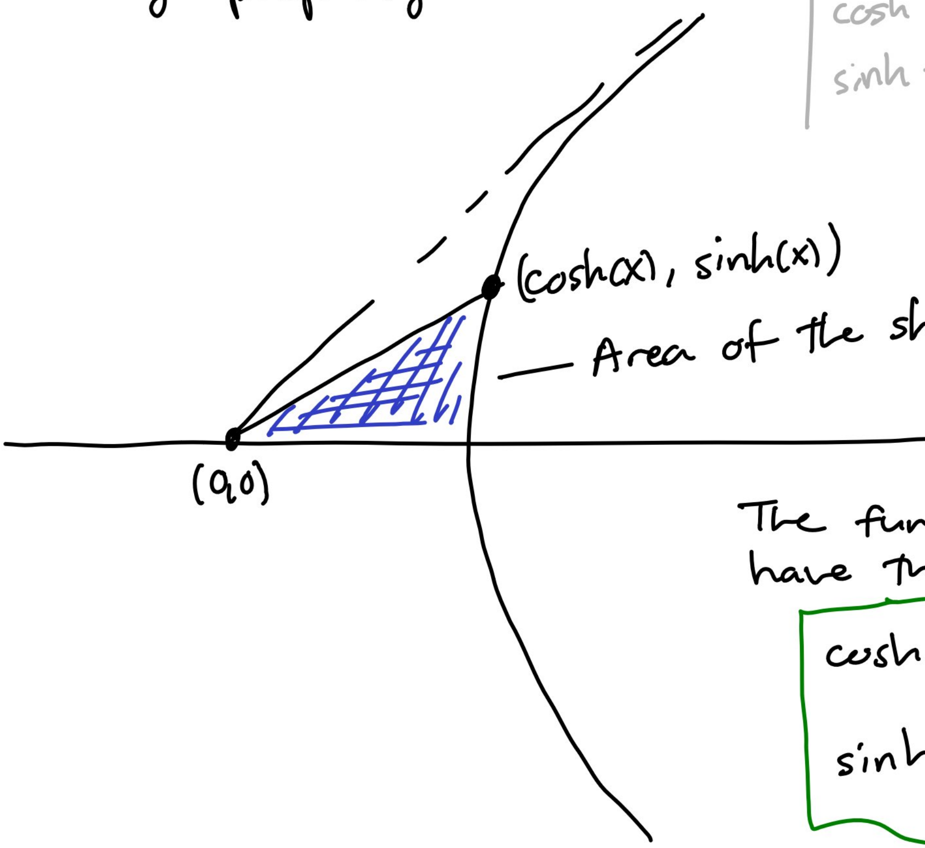


$x^2-y^2=1$

Parametrized by	Parametrized by
$\theta \mapsto (\cos\theta, \sin\theta)$ $\theta \in [0, 2\pi)$	$x \mapsto (\cosh x, \sinh x)$, $x \in \mathbb{R}$

$\cosh(x)$ and $\sinh(x)$ are determined by the following property:

cosh = hyperbolic cosine
sinh = hyperbolic sine



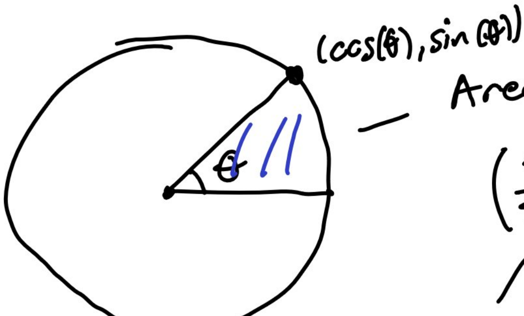
Area of the shaded region is $\frac{x}{2}$.

The functions that have this property are

$$\boxed{\begin{aligned}\cosh(x) &= \frac{e^x + e^{-x}}{2} \quad \text{and} \\ \sinh(x) &= \frac{e^x - e^{-x}}{2}.\end{aligned}}$$

Def. of cosh(x) and sinh(x).

This is motivated by a similar property of the usual cosine and sine:



Area of shaded region is

$$\underbrace{\left(\frac{\theta}{2\pi}\right)}_{\text{Portion of circumference}} \cdot \underbrace{\pi}_{\text{Area of unit circle}} = \frac{\theta}{2}$$

Useful properties of cosh and sinh:

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{e^x+e^{-x}}{2} = \cosh(x)$$

(cosh and sinh are the other's derivative).

Similar to $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$

Since cosh and sinh parametrize the unit hyperbola, we should expect that

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Indeed,

$$\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x}+2e^xe^{-x}+e^{-2x}}{4} - \frac{e^{2x}-2e^xe^{-x}+e^{-2x}}{4} = 1.$$

Now we come to $\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Let $u(x) = \frac{dy}{dx}(x)$.

The equation may be written $\frac{du}{dx} = \sqrt{1+u^2}$.

This is first-order in u, and separable!

$$\frac{1}{\sqrt{1+u^2}}\frac{du}{dx} = k.$$

Integrating both sides,

$$\int \frac{du}{\sqrt{1+u^2}} = \int k \, dx$$

Let $u = \sinh(t)$

$du = \cosh(t) \, dt$

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\cosh(t) \, dt}{\sqrt{1+\sinh^2(t)}}$$

$$= \int \frac{\cosh(t) \, dt}{\sqrt{\cosh^2(t)}} = \int dt = t$$

So, we have

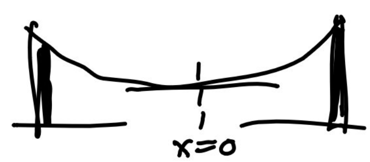
$$t = kx + c.$$

Taking sinh of both sides,

$$u = \sinh(t) = \sinh(kx + c)$$

Since we chose a coordinate system with the bottom point of the cable at $x=0$, we have the initial condition

$$u(0) = \frac{dy}{dx}(0) = 0$$



We determine c:

$$0 = u(0) = \sinh(0+c) = \sinh(c) = \frac{e^c - e^{-c}}{2}.$$

$e^c = e^{-c}$ iff $c = -c$ iff $c = 0$.

So $\frac{dy}{dx}(x) = u(x) = \sinh(kx)$.

Integrating with respect to x, we get

$$y(x) = \frac{1}{k} \cosh(kx) + D$$

The constant of integration D is related to the y-coordinate of the bottom point of the cable in our coordinate system (so far, we have only set $x=0$, y is arbitrary).

$$y\text{-coordinate of bottom point} = y(0) = \frac{1}{k} \cosh(0) + D$$

$$= \frac{1}{k} \frac{e^0 + e^{-0}}{2} + D$$

$$= \frac{1}{k} + D$$

If we choose the coordinate system so that the bottom point has y-coordinate $\frac{1}{k}$, we have $D=0$, and obtain the solution

$$y(x) = \frac{1}{k} \cosh(kx), \quad -L < x < L.$$