Mthe 237
Lecture 02
Sept. 13, 2017

Ordinary eqs. depend on one independent variable, partial eqs on ≥ 1

e.g. $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$ is a partial diff. eq.

<u>Def.</u> An (ordinary) differential equation is an equation of the form

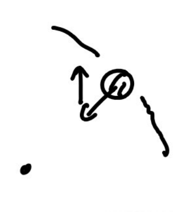
$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^r y}{dx^r}\right) = 0, \qquad (*)$$

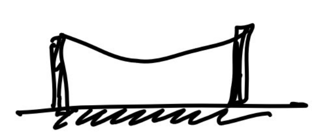
Here $F: \mathbb{R}^{r+2} \longrightarrow \mathbb{R}$ is a function (of $(r+2)$ variables)
(In practice, F is frequently defined by a formula.)

The highest derivative that appears in $(*)$ is called the <u>order</u> of the equation.

<u>Examples:</u>

	<u>Order</u>
$\frac{dy}{dx} - y = 0$	1
$\left(\frac{dy}{dx}\right)^2 - x = 0$	1
$\frac{d^2y}{dx^2} + y = 0$	2
$\frac{dy}{dx} - \frac{y - k\sqrt{x^2+y^2}}{x} = 0, \ x \neq 0$	1
$\frac{d^2y}{dx^2} - k\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0$	2





A function $\phi : I \longrightarrow \mathbb{R}$ is said to be a solution to $(*)$ on an interval $I \subset \mathbb{R}$ if

$$F\left(x, \phi(x), \frac{d\phi}{dx}(x), \ldots, \frac{d^r\phi}{dx}(x) \right) = 0$$

for all $x \in I$.

(For this expression to make sense, ϕ must be at least r-times differentiable)

Later, we shall often use the notation $y(x)$ instead of $\phi(x)$.

For the examples above,

$$\frac{dy}{dx} - y = 0 \qquad \phi(x) = Ce^x \quad \left(= C\exp(x) \text{ alternative notation} \right)$$

$I = \mathbb{R}$

C is a real constant

is a family of solutions

Check: $\frac{d\phi}{dx}(x) = Ce^x = \phi(x)$, so $\frac{d\phi}{dx}(x) - \phi(x) = 0$.

To solve $\left(\frac{dy}{dx}\right)^2 - x = 0$, we solve for $\frac{dy}{dx}$ first, getting two equations:

$$\left(\frac{dy}{dx}\right)^2 = x$$

$\frac{dy}{dx} = x^{1/2}$	$\frac{dy}{dx} = -x^{1/2}$
$\phi(x) = \frac{2}{3}x^{3/2} + C$	$\phi(x) = -\frac{2}{3}x^{3/2} + C$

We combine these two families of solutions by writing

$$\phi(x) = C \pm \frac{2}{3} x^{3/2}, \quad \underbrace{x>0}_{}$$

$$(I = \{x \in \mathbb{R} : x > 0\})$$

This procedure is typical. In practice, to solve

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^r y}{dx^r}\right) = 0,$$

we solve for $\frac{d^r y}{dx^r}$ in the other terms, obtaining a number of equations of form

$$\frac{d^r y}{dx^r} = G\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{r-1} y}{dx^{r-1}}\right),$$

which then yield solutions to the original differential equation

(In the example,

$$F\left(x, \frac{dy}{dx}\right) = \left(\frac{dy}{dx}\right)^2 - x \quad \text{is solved to get}$$

$$\frac{dy}{dx} = G_1(x) = x^{1/2} \quad \text{and}$$

$$\frac{dy}{dx} = G_2(x) = -x^{1/2}$$

).

For this reason, we frequently start with equations in the form

$$\frac{d^r y}{dx^r} = G\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{r-1} y}{dx^{r-1}}\right)$$

(assuming we have already solved for $\frac{d^r y}{dx^r}$).

Solutions to $\frac{d^2y}{dx^2} + y = 0$

Take the form $A\sin(x) + B\cos(x)$, $x \in \mathbb{R}$.

<u>Check</u>: $\phi(x) = A\sin(x) + B\cos(x)$

$$\phi'(x) = A\cos(x) - B\sin(x)$$

$$\phi''(x) = -A\sin(x) - B\cos(x)$$

$$= -\phi(x)$$

So, $\phi''(x) + \phi(x) = 0$ for all $x \in \mathbb{R}$.

We will learn how to solve the other two equations (and obtain the above solutions systematically) this term!

As seen in these examples, solutions to a differential equation typically come in families that depend on a number of free parameters.

As a rule-of-thumb:

$$\text{Number of Parameters} = \text{Order of the diff. eq.}$$

To pick out solutions, one specifies additional data, called <u>initial conditions</u>.

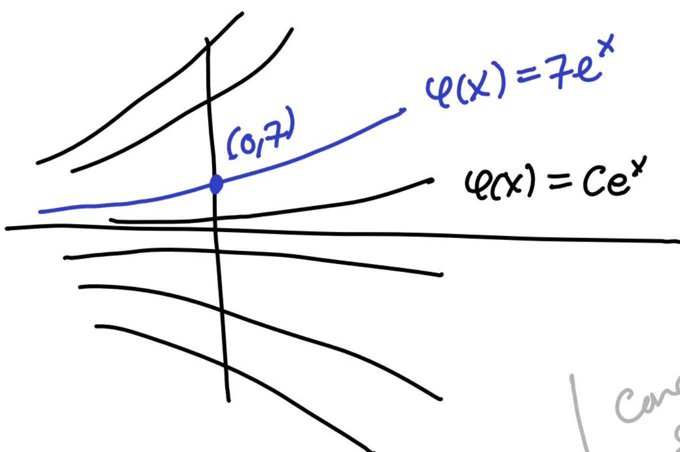
Examples:

- $\dfrac{dy}{dx} = y$, $\quad y(0) = 7.$

$\phi(x) = Ce^x$ is a solution. Plug in $x=0$,

$7 = \phi(0) = Ce^0 = C \cdot 1 = C.$

So $\phi(x) = 7e^x$ is a solution satisfying the initial condition.



Conditions of this type are sometimes called boundary conditions instead of initial conditions.

- $\dfrac{d^2y}{dx^2} + y = 0$,

$y(0) = 0$,
$y(\frac{\pi}{2}) = 1.$

$\varphi(x) = A\sin(x) + B\cos(x)$

$\varphi(0) = A\sin(0) + B\cos(0)$
$= A \cdot 0 + B \cdot 1 = B$
$\Rightarrow B = 0$

$\varphi(\frac{\pi}{2}) = A\sin(\frac{\pi}{2}) + 0 \cdot \cos(\frac{\pi}{2})$
$= A \cdot 1 = A \Rightarrow A = 1$

$\varphi(x) = \sin(x)$ is the sol.

$y(0) = 1$,
$y'(0) = 1.$

$\varphi(0) = A\sin(0) + B\cos(0)$
$= B \Rightarrow B = 1.$

$\varphi'(x) = A\cos(x) - 1 \cdot \sin(x)$

$\varphi'(0) = A\cos(0) - 1\sin(0)$
$= A \Rightarrow A = 1$

$\varphi(x) = \sin(x) + \cos(x)$ is the sol.

The number of initial conditions should be equal to the number of parameters to pick out a single solution (that no longer depends on parameters).

Initial conditions for the drone:

Picking a point on the trajectory should determine the trajectory (from our mechanics intuition)

An initial condition of the kind

$$y(x_0) = y_0$$

specifies a point on the trajectory.

(Note the drone eq. is order 1, so should depend on one parameter)

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Many things are not clear:

- Do solutions exist?
- How do we know we found all solutions?
- Do initial conditions uniquely determine solutions?

Will address these questions later.

First two weeks: techniques for solving first-order DEs

$$F\left(x, y, \frac{dy}{dx}\right) = 0.$$

Unfortunately, there is no method that works in general.

One of the most useful types:

Separable Equations:

Equations that can be put into the form

$$n(y)\,\frac{dy}{dx} = m(x),$$

for some functions m, n of a single variable.

Algorithm for Separable Equations:

① Bring to form $n(y)\,\frac{dy}{dx} = m(x)$.

② Solutions will be given implicitly by

$$\int n(y)\,dy = \int m(x)\,dx$$

③ When possible, solve for y as function of x.

Example: $\frac{dy}{dx} = ky$, k a real constant.

① $$\frac{1}{y}\cdot\frac{dy}{dx} = k \quad \left(m(x) = k, \quad n(y) = \frac{1}{y} \right)$$

② $$\int \frac{dy}{y} = \int k\,dx$$

$$\ln(|y|) = kx + C$$

$$|y| = e^{kx+C} = e^{C}e^{kx} = \tilde{C}e^{kx}$$

$\tilde{C} = e^{C} > 0$

$|y| = \tilde{c}e^{kx}$ implicitly describes two functions:

$$y(x) = \tilde{c}e^{kx} \quad \text{and} \quad y(x) = -\tilde{c}e^{kx}$$

We can describe both families uniformly by writing

$$y(x) = \tilde{C}e^{kx}, \qquad \tilde{C} \neq 0 \ \left(\text{could be negative}\right)$$

Finally, dividing by y in step ① lost the solution $y=0$. Putting this back in, we have

$$y(x) = Ce^{kx}, \quad C \in \mathbb{R}, \quad x \in \mathbb{R}.$$