Mthe 237
Lecture 01
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Instructor: Ilia SMIRNOV
Office : Jeffery Hall 111
Email : 9is15@queensu.ca

(Optional) Textbook:
Tenenbaum & Pollard. Ordinary Differential Equations. Dover.

Another commonly used textbook:
Boyce & DiPrima. Elementary Differential Equations. Wiley.

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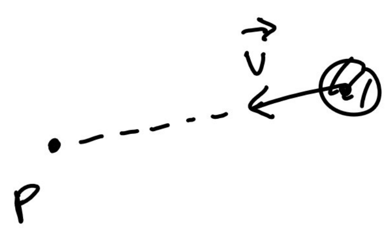
We started the course by deriving two differential equations that solve interesting problems, for some motivation.

<u>Problem 1</u> Suppose we want to study the trajectory of a drone that always moves toward a fixed point P, with constant speed v.



$|\vec{v}| = v.$

In absence of other complications, the trajectory is simple: the drone moves along a line toward P.

To make the problem more interesting, introduce a constant wind of speed w, from North to South.

What will be the trajectory?

<u>Aside</u>: The trajectory of a particle may be described parametrically by two differentiable functions $x(t), y(t)$, $t \in I$.

Position at time t is described by $(x(t), y(t))$.
Velocity at time t is $(\dot{x}(t), \dot{y}(t))$

$$\left(\dot{x}(t) = \frac{dx}{dt}(t) = x'(t) \right.$$ are all common notations for derivative of x with respect to t)

Newton Leibniz Lagrange

Suppose that the trajectory lies on the graph of a function $y(x)$. By the chain rule, we have

$$\frac{dy}{dt}(t) = \frac{dy}{dx}(x(t))\,\frac{dx}{dt}(t), \text{ so that}$$

$$\frac{dy}{dx}(x(t)) = \frac{dy}{dt}(t) \Big/ \frac{dx}{dt}(t) \quad \text{or} \quad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}.$$

(whenever $\dot{x} \neq 0$)

Ex. The top half of the unit circle $x^2+y^2=1$ may be parametrized by

$$t \longmapsto (\cos(t), \sin(t)), \quad t \in (0,\pi).$$

The trajectory lies on the graph of the function $y(x) = \sqrt{1-x^2}$ $(-1 < x < 1)$.

We can compute that $\frac{dy}{dx}(x) = -\frac{x}{\sqrt{1-x^2}}$,

so that

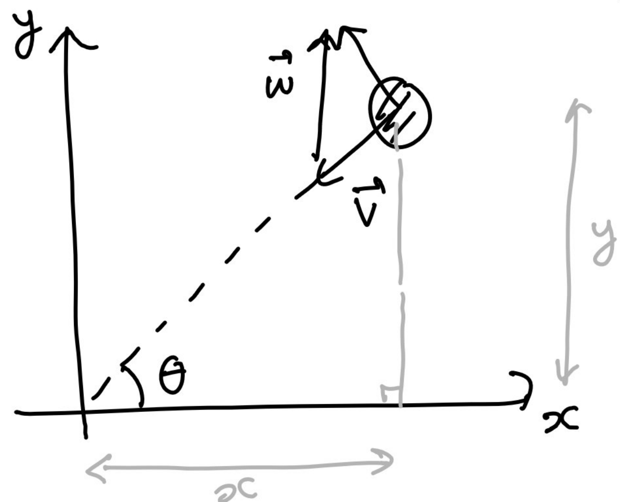
$$\frac{dy}{dx}(x(t)) = \frac{dy}{dx}(\cos(t)) = -\frac{\cos(t)}{\sqrt{1-\cos^2(t)}}$$

$$= -\frac{\cos(t)}{\sin(t)}$$

$$= \frac{\cos(t)}{-\sin(t)} = \frac{\dot{y}(t)}{\dot{x}(t)}$$

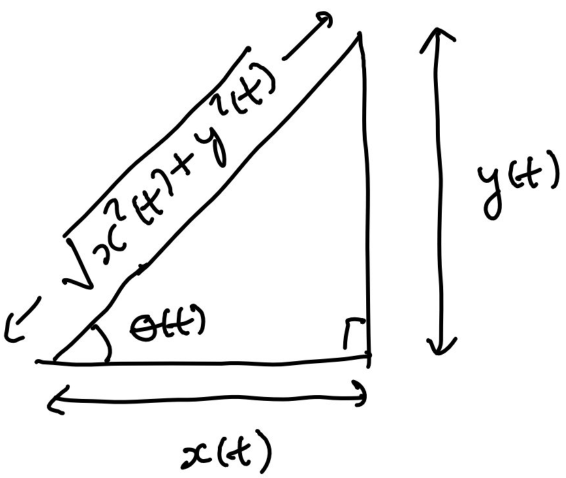
This verifies the relation above for this example.

Coming back to the drone, the velocity of the drone relative to ground is the vector sum of $\vec{v}$ and $\vec{w}$. Choosing coordinates with P at $(0,0)$, we have



$$\begin{cases} \dot{x}(t) = -v\cos\theta(t) \\ \dot{y}(t) = -v\sin\theta(t) + w \end{cases}$$

From the right triangle



we have

$$\cos\theta(t) = \frac{x(t)}{\sqrt{x^2(t)+y^2(t)}}$$

$$\sin\theta(t) = \frac{y(t)}{\sqrt{x^2(t)+y^2(t)}}$$

Using the relation $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ derived above,

$$\frac{dy}{dx}(x(t)) = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{-v\sin\theta(t)+w}{-v\cos\theta(t)}$$

$$= \frac{-v\frac{y(t)}{\sqrt{x^2(t)+y^2(t)}} + w}{-v\frac{x(t)}{\sqrt{x^2(t)+y^2(t)}}}$$

$$= \frac{-v\,y(t) + w\sqrt{x^2(t)+y^2(t)}}{-v\,x(t)}$$

Letting $k = \frac{w}{v}$, this is equal to

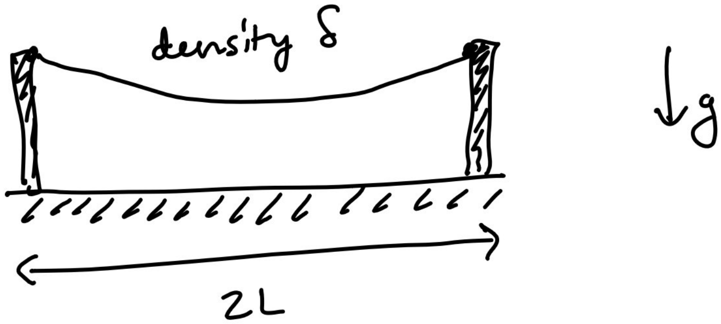
$$\frac{-y(t) + k\sqrt{x^2(t)+y^2(t)}}{-x(t)}.$$

Therefore, the function $y(x)$ describing the trajectory of the drone satisfies the diff. eq.

$$\boxed{\frac{dy}{dx} = \frac{-y + k\sqrt{x^2+y^2}}{-x}, \qquad x \neq 0}$$

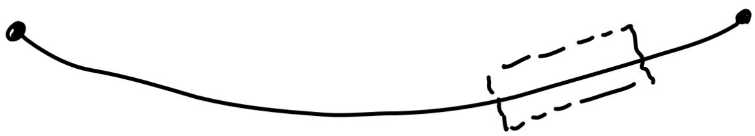
That we may call the drone equation.

<u>Problem 2.</u> What is the shape of a cable, wire or chain that hangs under its own weight from two supports (of equal height)?



We look for a function $y(x)$ whose graph describes the shape.

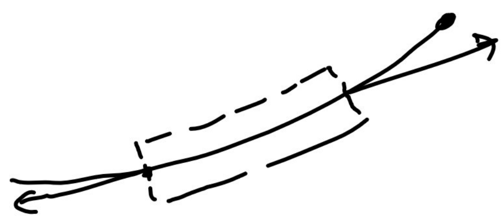
Let's (mentally) isolate a piece of the cable.

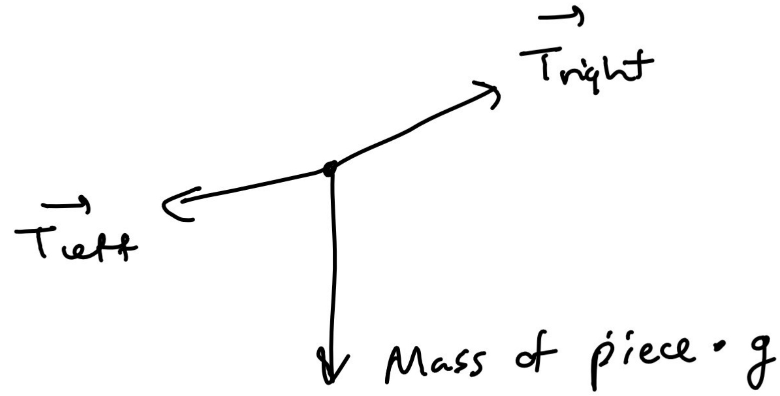


What will be the force on this isolated piece?

The internal forces will cancel out (Newton's third law)

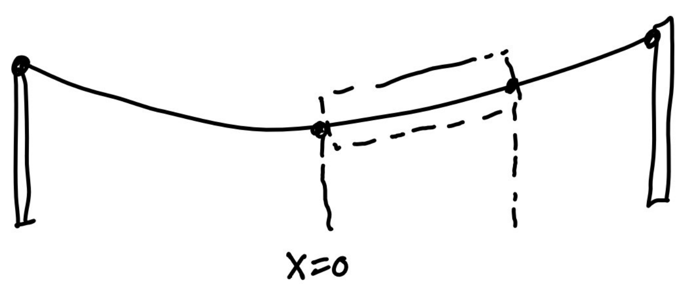
The neighbouring parts of the cable will cause two forces of tension, directed <u>tangent</u> to the cable.

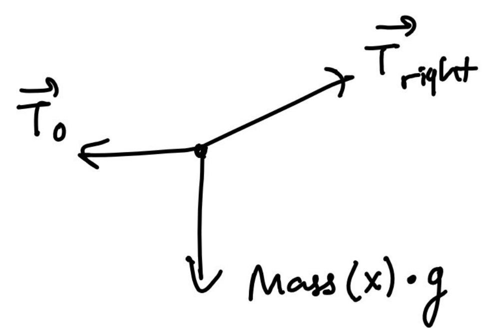




Fix the left endpoint at the bottom of the cable. Choose coordinates with this point at $x=0$.

The left tension will be horizontal.

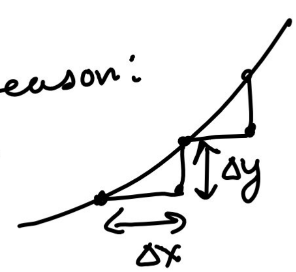




How to find the mass:

$$\int_0^x \delta \sqrt{1+y'(t)^2}\, dt$$

Reason:

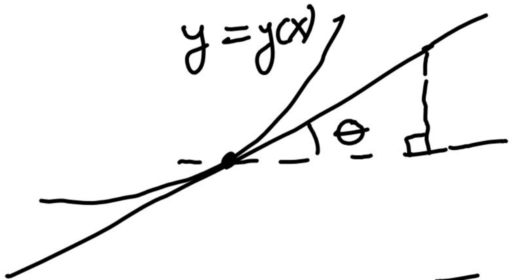


Break cable up into small triangles

$$\text{Mass} \approx \sum \delta \sqrt{\Delta x^2 + \Delta y^2} = \sum \delta \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$$\xrightarrow{\Delta x \to 0} \int \delta \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

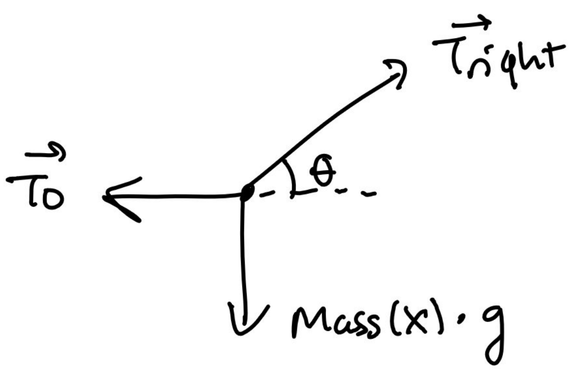
How to find the angle the tangent to $y(x)$ makes with the horizontal:



Slope of tangent $= \frac{\text{rise}}{\text{run}} = \tan\theta$

Therefore, $\frac{dy}{dx} = \tan(\theta)$

The cable is in stable equilibrium, so forces balance.



$|\vec{T_0}| = \cos\theta\,|\vec{T}_{right}(x)|$, and

$\text{Mass}(x)\cdot g = \sin\theta\,|\vec{T}_{right}(x)|$

Dividing the two expressions, we obtain

$$\frac{\text{Mass}(x)\cdot g}{|\vec{T_0}|} = \frac{\sin\theta\,|\vec{T}_{right}(x)|}{\cos\theta\,|\vec{T}_{right}(x)|} = \tan\theta = \frac{dy}{dx}$$

So $$\frac{dy}{dx} = \frac{g\int_0^x \delta\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dt}{|\vec{T_0}|}.$$

Differentiating (using Fund. Thm. of Calculus)

$$\frac{d^2y}{dx^2} = \frac{g\delta}{|\vec{T_0}|}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

The solutions to the equation

$$\boxed{\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

describe the shape of the cable.

The curve describing the shape of the cable is called the catenary (after "catena", which is Latin for "chain"). So we may call this the catenary equation.