Problem 1 (10 points)**.** Solve the following differential equation. You may leave your solution in implicit form.

$$(x^2 - x + y^2) + (2xy - e^{-y})\frac{dy}{dx} = 0, \quad y(0) = 1.$$

Solution. Write

$$\begin{aligned} M(x,y) &= x^2 - x + y^2 \quad \text{and} \\ N(x,y) &= 2xy - e^{-y}. \end{aligned}$$

We have

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x},$$

therefore, since it is also true that $M(x,y)$, $N(x,y)$, $\partial M/\partial y(x,y)$ and $\partial N/\partial x(x,y)$ are continuous for all $(x,y) \in \mathbb{R}^2$, the equation is exact by the criterion proved in class.

We look for a function $G(x,y)$ such that $\partial G/\partial x = M$ and $\partial G/\partial y = N$. Implicit solutions of the differential equation will then be given by $G(x,y) = 0$.

Integrating $M(x,y)$ with respect to x, we get

$$\frac{1}{3}x^3 - \frac{1}{2}x^2 + xy^2 + h(y)$$

for some yet-undetermined function h of y. Taking the partial of this result with respect to y, we get

$$2xy + h'(y).$$

Matching this with $N(x,y)$, we see that we need

$$h'(y) = -e^{-y},$$

so that

$$h(y) = e^{-y} + C.$$

Thus, we can take

$$G(x,y) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + xy^2 + e^{-y} + C.$$

Finally, the initial condition determines C:

$$0 = G(0,1) = 1/e + C,$$

so that $C = -1/e$, and the implicit solution satisfying the initial condition is given by

$$\frac{x^3}{3} - \frac{x^2}{2} + xy^2 + e^{-y} - e^{-1} = 0,$$

or equivalently, clearing denominators,

$$2x^3 - 3x^2 + 6xy^2 + 6e^{-y} = 6e^{-1}.$$

Alternative solution. Integrating $N(x,y)$ with respect to y, we get

$$xy^2 + e^{-y} + h(x),$$

for some yet-undetermined function h of x.

Taking partial derivative with respect to x, we get

$$y^2 + h'(x).$$

Matching this with $M(x,y)$, we see that

$$h'(x) = x^2 - x,$$

so that

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + C.$$

Thus, we again have

$$G(x,y) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + xy^2 + e^{-y} + C,$$

and we can determine the constant C from the initial condition as in the previous solution.

Problem 2 (5+10=15 points). i) Show that

$$\int \frac{ds}{s\ln s} = \ln(\ln(s)) + C.$$

ii) Solve the differential equation

$$(x^2 - 1)\frac{dy}{dx} = 2xy\ln(y), \quad y(0) = e^{-1}.$$

Solution. i) Making the substitution $u = \ln(s)$, $du = ds/s$, the integral becomes

$$\int \frac{du}{u} = \ln(u) + C = \ln(\ln(s)) + C.$$

ii) This equation is separable. Separating variables, we have

$$\frac{1}{y\ln(y)}\frac{dy}{dx} = \frac{2x}{x^2-1}, \qquad y \neq 0,\ y \neq 1,\ x \neq \pm 1.$$

Integrating both sides, we get

$$\int \frac{dy}{y\ln(y)} = \int \frac{2x}{x^2-1}\,dx.$$

By part i), the left side is $\ln(|\ln(y)|)$, and the right side is $\ln(|x^2-1|)+C$ (for example, by substituting $u = x^2 - 1,\ du = 2x\,dx$).

Taking exponentials of both sides, we get

$$\ln(y) = C\left|x^2 - 1\right|,$$

where C is allowed to be negative.

Imposing the initial condition, we get

$$\ln(e^{-1}) = C,$$

so that $C = -1$. Finally, taking another exponential, we find

$$y = \exp(-\left|x^2 - 1\right|),$$

To match the initial condition and restrictions that came up in the problem, we need to take $-1 < x < 1$ ($x \neq \pm 1$, and the domain of the solution is a connected interval that contains the x-coordinate of the initial condition, which is equal to 0 in this problem). In this domain, $x^2 - 1 < 0$, so $-|x^2-1| = x^2 - 1$. Since $0 < \exp(x^2-1) < 1$ for such x, there are no additional restrictions on the domain coming from the conditions on y. Therefore, the solution is

$$y(x) = \exp(x^2 - 1), \quad -1 < x < 1.$$

Problem 3 (5+15+5=25 points)**.** Let a be a nonzero real number. Consider the differential equation

$$\frac{dy}{dx} = y^2 + (\pi a/2)^2, \quad y(0) = 0. \tag{1}$$

i) What is the strongest conclusion that can be made regarding solutions of equation (1) using the Existence and Uniqueness Theorem for First Order Differential Equations?

   (Is a solution certain to exist in some open interval containing 0? If so, is a solution certain to be unique in some open interval containing 0?)

ii) Find a function ϕ of x that solves the differential equation (1). What is the largest domain over which ϕ is defined and differentiable (the answer will depend on the number a)? Denote this domain by I_a.

   (The following integral may be useful: for any nonzero $\alpha \in \mathbb{R}$, $\int \frac{ds}{s^2+\alpha^2} = \frac{1}{\alpha}\arctan\left(\frac{s}{\alpha}\right) + C$.)

iii) Recall that the *length* of an open interval $(c, d) = \{x \in \mathbb{R} : c < x < d\}$ is defined to be $d - c$. For example, the length of $(3, 7)$ is 4 and the length of $(-2, 1)$ is 3.

   What is the length of the domain I_a found in part ii)? Find a value of a so that the length of I_a is less than or equal to $\frac{1}{1000}$.

Solution. i) The differential equation we are working with is

$$\frac{dy}{dx} = F(x,y), \qquad F(x,y) = y^2 + (\pi a/2)^2, \qquad y(0) = 0. \tag{2}$$

(Despite the fact that $F(x,y)$ does not depend on x, it is a function of two variables.)

Because $F(x,y)$ is a polynomial, it is continuous at all $(x,y) \in \mathbb{R}^2$. Take $R = \mathbb{R}^2$. Since F is continuous over R, and R contains the point $(0,0)$ in its interior, we can conclude by the Existence part of the Existence and Uniqueness Theorem that there exists some interval I containing 0, and a solution $\phi(x)\colon I \to \mathbb{R}$ of equation (2).

For the Uniqueness part of the theorem, we need to compute

$$\frac{\partial F}{\partial y} = 2y.$$

$\partial F/\partial y$ is again a polynomial, hence continuous at all $(x,y) \in \mathbb{R}^2$. Again, we can take R to be all of $\mathbb{R}^2$, and conclude that there exists some open interval $I' \subseteq I$ about $x = 0$, such that if $\phi_1(x)$ and $\phi_2(x)$ are two solutions of (2) over I' (in particular, satisfying the initial condition $\phi_1(0) = \phi_2(0) = 0$), we have $\phi_1(x) = \phi_2(x)$ for all $x \in I'$.

In summary, we can conclude by the Existence and Uniqueness Theorem that a unique solution exists with domain some (possibly small) open interval I containing $x = 0$.

ii) The equation is separable. Separating variables, we have

$$\frac{1}{y^2 + (\pi a/2)^2}\frac{dy}{dx} = 1,$$

so that, integrating both sides,

$$\int \frac{dy}{y^2 + (\pi a/2)^2} = \int 1\,dx.$$

We get

$$\frac{1}{\pi a/2}\arctan\left(\frac{y}{\pi a/2}\right) = x + C.$$

Using the initial condition,

$$\frac{1}{\pi a/2}\arctan\left(\frac{0}{\pi a/2}\right) = 0 + C.$$

Since $\arctan(0) = 0$, we conclude that $C = 0$.

Now, solving for y as a function of x, we get

$$\arctan\left(\frac{y}{\pi a/2}\right) = \frac{\pi a x}{2},$$

so that

$$\phi(x) = \frac{\pi a}{2}\tan\left(\frac{\pi a x}{2}\right).$$

The function tan has infinite discontinuities at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. Since solutions of differential equations are differentiable, hence continuous, this restricts the domain of the solution. The domain should be an open interval that contains $x = 0$ (the x-coordinate of the initial condition). Therefore, the domain is all of x that satisfy the two inequalities

$$-\frac{\pi}{2} < \frac{\pi a x}{2} < \frac{\pi}{2}.$$

Dividing through by $\pi a/2$, this gives

$$-\frac{1}{a} < x < \frac{1}{a}.$$

This is then the domain of the solution:

$$I_a = \left\{x \in \mathbb{R} : -\frac{1}{a} < x < \frac{1}{a}\right\} = \left(-\frac{1}{a}, \frac{1}{a}\right).$$

iii) The length of I_a is

$$\frac{1}{a} - \left(-\frac{1}{a}\right) = \frac{2}{a}.$$

We need to find a so that

$$\frac{2}{a} \leq \frac{1}{1000}.$$

Any

$$a \geq 2000$$

would satisfy this inequality. For instance, we can take $a = 2017$.

Remark. This problem illustrates the following interesting point: even though the function $F(x, y)$ and its partial $\partial F/\partial y(x, y)$ are continuous over all of $\mathbb{R}^2$, the interval of existence of the solution can be made arbitrarily small by making a sufficiently large.

This is in sharp distinction with the case of linear differential equations, whose solutions exist (and are unique) over the entire interval where we have continuity of the coefficients in the equation

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

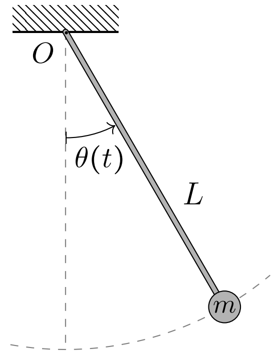
(The equation

$$\frac{dy}{dx} = y^2 + (\pi a/2)^2$$

is nonlinear because of the y^2 term.)

Problem 4 (10+5+5+10+20=50 points)**.** Consider an example of a simple pendulum: it consists of a rigid, but weightless, linear rod of length L, one of whose ends is attached to a fixed point O, and a mass m that is attached to the free end of the rod.

Denote the angle the rod makes with the vertical line passing through the point O at time t by $\theta(t)$. The motion of the pendulum is completely described by $\theta(t)$.



Suppose that gravity is uniform, and points down. One can show that when the angle $\theta(t)$ remains close to 0 throughout the motion, so that we can make the *small angle approximation* $\sin(\theta) \approx \theta$, to a good approximation $\theta(t)$ satisfies the differential equation

$$mL^2 \frac{d^2\theta}{dt^2} + dL\frac{d\theta}{dt} + mgL\theta = 0,$$

where $d > 0$ is a damping constant, and g is the gravitational constant.

i) In terms of the constants m, L, d, and g, characterize when the pendulum is underdamped, critically damped, and overdamped. Briefly describe a typical motion of the pendulum in each of these three cases.

Suppose that $d/mL = 6$ and $g/L = 9$. In the absence of a driving torque, the motion of the pendulum is then described by the equation

$$\frac{d^2\theta}{dt^2} + 6\frac{d\theta}{dt} + 9\theta = 0. \tag{3}$$

ii) Find the roots of the characteristic polynomial $\chi(z)$ of equation (3).

iii) Find a basis $\{\phi_1, \phi_2\}$ of the space of solutions of equation (3).

iv) Compute the Wronskian $W(\phi_1, \phi_2)(t)$ of ϕ_1 and ϕ_2. Using the result, verify that ϕ_1 and ϕ_2 are linearly independent.

If a clockwise torque of magnitude $te^{-3t}mL^2$ is applied to the pendulum, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + 6\frac{d\theta}{dt} + 9\theta = te^{-3t}. \tag{4}$$

v) Solve equation (4) subject to the initial conditions

$$\theta(0) = 0, \quad \frac{d\theta}{dt}(0) = 0,$$

using your preferred method.

Solution. i) The types of motion are determined by the types of roots of the characteristic polynomial of the differential equation. Dividing through by mL^2, we get

$$\frac{d^2\theta}{dt^2} + \frac{d}{mL}\frac{d\theta}{dt} + \frac{g}{L}\theta = 0,$$

which has the characteristic polynomial

$$\chi(z) = z^2 + \frac{d}{mL}z + \frac{g}{L}.$$

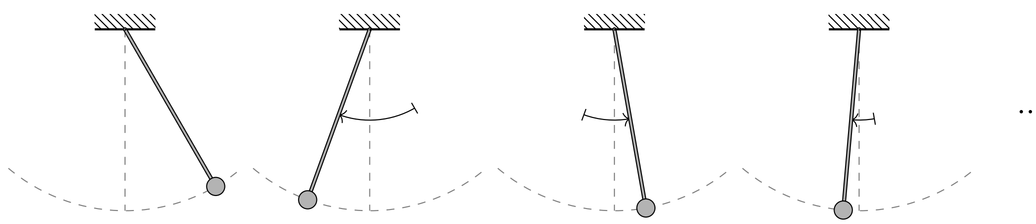
Recall that the discriminant of a quadratic polynomial $az^2 + bz + c$ is by definition $\Delta = b^2 - 4ac$. For the characteristic polynomial $\chi(z)$, the discriminant is

$$\left(\frac{d}{mL}\right)^2 - 4\frac{g}{L} = \frac{d^2 - 4m^2Lg}{m^2L^2}.$$

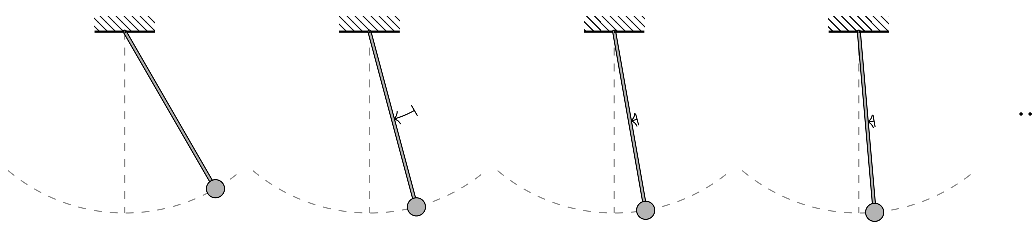
The sign of the discriminant is the same as the sign of the numerator, so we focus on

$$d^2 - 4m^2Lg.$$

- If $d^2 - 4m^2Lg < 0$ (equivalently, $d < 2m\sqrt{Lg}$), then the characteristic polynomial has *two conjugate complex roots*. The pendulum is said to be *underdamped*, and a typical motion is a sinusoidal oscillation with an exponentially decaying amplitude.



- If $d^2 - 4m^2Lg > 0$ (equivalently, $d > 2m\sqrt{Lg}$), then the characteristic polynomial has *two distinct real roots*. The pendulum is said to be *overdamped*, and a typical motion either decays towards the rest position, passing the rest position at most once. There is no sinusoidal oscillation about the rest position.



- If $d^2 - 4m^2Lg = 0$ (equivalently, $d = 2m\sqrt{Lg}$), then the characteristic polynomial has *a repeated real root*. The pendulum is said to be *critically damped*. The motion is similar to the overdamped case, but returns to the rest position in the least time among all overdamped motions.

ii) The characteristic polynomial is

$$\chi(z) = z^2 + 6z + 9 = (z+3)^2.$$

There is a double real root at $z = -3$.

iii) By the general procedure, the basis of solutions is

$$\left\{\phi_1(t) = e^{-3t},\ \phi_2(t) = te^{-3t}\right\}.$$

iv) Computing the derivatives of ϕ_1 and ϕ_2, we have

$$\begin{aligned}(e^{-3t})' &= -3e^{-3t}\\ (t\,e^{-3t})' &= e^{-3t} - 3te^{-3t} = (1-3t)\,e^{-3t}.\end{aligned}$$

Therefore, the Wronskian is

$$W(\phi_1, \phi_2)(t) = \det\begin{pmatrix} e^{-3t} & t\,e^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t}\end{pmatrix} = (1-3t)e^{-6t} + 3te^{-6t} = e^{-6t}.$$

At $t = 0$, we have

$$W(\phi_1, \phi_2)(0) = e^0 = 1 \neq 0,$$

so we can conclude that ϕ_1 and ϕ_2 are indeed linearly independent (we have proven that the functions given by the procedure for writing down a basis of solutions are indeed linearly independent in class, but it is nice to have an independent check).

Finally, we see that $W(\phi_1, \phi_2)(t) \neq 0$ for all t, as should be the case by Abel's theorem.

v) *Annihilator method.* We need to find an annihilator of te^{-3t}. But te^{-3t} is actually one of the basis elements of the solutions of

$$\left(\frac{d}{dt} + 3\right)^2 y = 0$$

we found above (namely, ϕ_2). Hence,

$$\left(\frac{d}{dt} + 3\right)^2$$

is an annihilator of te^{-3t}.

Applying the annihilator to both sides of eq. (4), we obtain the linear homogeneous equation

$$\left(\frac{d}{dt} + 3\right)^2 \left(\frac{d}{dt} + 3\right)^2 y = \left(\frac{d}{dt} + 3\right)^4 y = 0. \tag{5}$$

The basis of solutions of eq. (5) is

$$\left\{e^{-3t},\ te^{-3t},\ t^2e^{-3t},\ t^3e^{-3t}\right\}.$$

The first two are solutions of the associated homogeneous equation, hence will not help in finding the particular solution. So, take

$$\phi_p = At^2e^{-3t} + Bt^3e^{-3t}.$$

Differentiating, we have

$$\begin{aligned}
\phi_p' &= A\left(2te^{-3t} - 3t^2e^{-3t}\right) + B\left(3t^2e^{-3t} - 3t^3e^{-3t}\right) \\
&= 2A\,te^{-3t} - (3A - 3B)\,t^2e^{-3t} - 3B\,t^3e^{-3t} \\
\phi_p'' &= 2A\left(e^{-3t} - 3te^{-3t}\right) - (3A - 3B)\left(2te^{-3t} - 3t^2e^{-3t}\right) - 3B\left(3t^2e^{-3t} - 3t^3e^{-3t}\right) \\
&= 2A\,e^{-3t} + (-6A - 6A + 6B)\,te^{-3t} + (9A - 9B - 9B)\,t^2e^{-3t} + 9B\,t^3e^{-3t} \\
&= 2A\,e^{-3t} + (-12A + 6B)\,te^{-3t} + (9A - 18B)\,t^2e^{-3t} + 9Bt^3e^{-3t}.
\end{aligned}$$

Now, computing $\phi_p'' + 6\phi_p' + 9\phi_p$, we have

$$\begin{aligned}
&2A\,e^{-3t} + (-12A + 6B)\,te^{-3t} + (9A - 18B)\,t^2e^{-3t} + 9B\,t^3e^{-3t} \\
&+6\left(2A\,te^{-3t} - (3A - 3B)\,t^2e^{-3t} - 3B\,t^3e^{-3t}\right) \\
&+9(A\,t^2e^{-3t} + B\,t^3e^{-3t}) \\
&= 2A\,e^{-3t} + (-12A + 6B + 12A)\,te^{-3t} + (9A - 18B - 18A + 18B + 9A)\,t^2e^{-3t} + (9B - 18B + 9B)\,t^3e^{-3t} \\
&= 2A\,e^{-3t} + 6B\,te^{-3t}.
\end{aligned}$$

For ϕ_p to be a solution of (4), we have to have $\phi_p'' + 6\phi_p' + 9\phi_p = te^{-3t}$. Comparing coefficients, we get the system of equations

$$\begin{aligned}
2A &= 0 \\
6B &= 1.
\end{aligned}$$

Therefore, $A = 0$, $B = \frac{1}{6}$ and

$$\phi_p = \frac{t^3e^{-3t}}{6}.$$

Now, the affine space of all solutions is

$$\phi_p + \{\text{Solutions of } \frac{d^2\theta}{dt} + 6\frac{d\theta}{dt} + 9\theta = 0\} = \frac{t^3e^{-3t}}{6} + Ce^{-3t} + Dte^{-3t}, \qquad C, D \in \mathbb{R}.$$

We determine C and D by the initial conditions. The derivative is

$$\frac{3t^2e^{-3t} - 3t^3e^{-3t}}{6} - 3Ce^{-3t} + De^{-3t} - 3Dte^{-3t}.$$

From $\theta(0) = 0$, we have

$$0 + Ce^0 + 0 = 0,$$

and from $\theta'(0) = 0$, we have

$$0 - 3Ce^0 + De^0 - 0 = 0.$$

Thus,

$$C = 0$$
$$-3C + D = 0$$

and $C = D = 0$.

The solution satisfying the initial conditions is

$$\phi(t) = \frac{t^3 e^{-3t}}{6}.$$

Variation of parameters. Take e^{-3t} and te^{-3t} as the solutions of the associated homogeneous equation (found above). The variation-of-parameters particular solution is given by

$$\phi_p = u_1\, e^{-3t} + u_2\, te^{-3t},$$

where u_1 and u_2 are functions whose derivatives satisfy the system of equations

$$\begin{pmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ te^{-3t} \end{pmatrix}.$$

We use Cramer's rule to solve the system of equations.

$$u_1' = \frac{\det \begin{pmatrix} 0 & te^{-3t} \\ te^{-3t} & (1-3t)e^{-3t} \end{pmatrix}}{W(e^{-3t}, te^{-3t})(t)} = \frac{-t^2 e^{-6t}}{e^{-6t}} = -t^2.$$

Therefore,

$$u_1 = \int -t^2\, dt = -\frac{t^3}{3}.$$

We have

$$u_2' = \frac{\det \begin{pmatrix} e^{-3t} & 0 \\ -3e^{-3t} & te^{-3t} \end{pmatrix}}{W(e^{-3t}, te^{-3t})(t)} = \frac{te^{-6t}}{e^{-6t}} = t.$$

Therefore,

$$u_2 = \int t\, dt = \frac{t^2}{2}.$$

Finally,

$$\phi_p = u_1 e^{-3t} + u_2 te^{-3t} = -\frac{t^3 e^{-3t}}{3} + \frac{t^3 e^{-3t}}{2} = \frac{t^3 e^{-3t}}{6}.$$

Now, proceed as in the undetermined coefficients solution, adding a solution to the associated homogeneous equation to satisfy the initial conditions (in this case, it turns out that the particular solution already satisfies the initial conditions).

Laplace transform. Taking the Laplace transform of both sides, we get

$$\mathscr{L}[\theta''](s) + 6\mathscr{L}[\theta'](s) + 9\mathscr{L}[\theta](s) = \mathscr{L}[te^{-3t}](s) = \frac{1}{(s+3)^2}.$$

Therefore, using the initial conditions $\theta(0) = \theta'(0) = 0$,

$$s^2\mathscr{L}[\theta](s) + 6s\mathscr{L}[\theta](s) + 9\mathscr{L}[\theta] = (s^2 + 6s + 9)\mathscr{L}[\theta](s) = (s+3)^2\mathscr{L}[\theta](s) = \frac{1}{(s+3)^2}.$$

Finally,

$$\mathscr{L}[\theta](s) = \frac{1}{(s+3)^4}.$$

From the transform

$$\mathscr{L}[t^3e^{-3t}](s) = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4},$$

we see that

$$\theta(t) = \frac{t^3e^{-3t}}{3!} = \frac{t^3e^{-3t}}{6}.$$