Problem 1 (10 points)**.** Solve the following differential equation. You may leave your solution in implicit form.

$$(x^2 - x + y^2) + (2xy - e^{-y})\frac{dy}{dx} = 0, \quad y(0) = 1.$$

Problem 2 (5+10=15 points)**.**

i) Show that

$$\int \frac{ds}{s \ln s} = \ln(\ln(s)) + C.$$

ii) Solve the differential equation

$$(x^2 - 1)\frac{dy}{dx} = 2xy\ln(y), \quad y(0) = e^{-1}.$$

Problem 3 (5+15+5=25 points)**.** Let a be a nonzero real number. Consider the differential equation

$$\frac{dy}{dx} = y^2 + (\pi a/2)^2, \quad y(0) = 0. \tag{1}$$

i) What is the strongest conclusion that can be made regarding solutions of equation (1) using the Existence and Uniqueness Theorem for First Order Differential Equations?

(Is a solution certain to exist in some open interval containing 0? If so, is a solution certain to be unique in some open interval containing 0?)

ii) Find a function ϕ of x that solves the differential equation (1). What is the largest domain over which ϕ is defined and differentiable (the answer will depend on the number a)? Denote this domain by I_a.

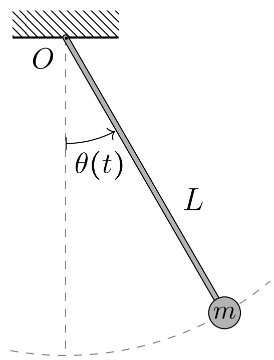
(The following integral may be useful: for any nonzero $\alpha \in \mathbb{R}$, $\int \frac{ds}{s^2 + \alpha^2} = \frac{1}{\alpha}\arctan\left(\frac{s}{\alpha}\right) + C$.)

iii) Recall that the *length* of an open interval $(c, d) = \{x \in \mathbb{R} : c < x < d\}$ is defined to be $d - c$. For example, the length of $(3, 7)$ is 4 and the length of $(-2, 1)$ is 3.

What is the length of the domain I_a found in part ii)? Find a value of a so that the length of I_a is less than or equal to $\frac{1}{1000}$.

Problem 4 (10+5+5+10+20=50 points)**.** Consider an example of a simple pendulum: it consists of a rigid, but weightless, linear rod of length L, one of whose ends is attached to a fixed point O, and a mass m that is attached to the free end of the rod.

Denote the angle the rod makes with the vertical line passing through the point O at time t by $\theta(t)$. The motion of the pendulum is completely described by $\theta(t)$.



Suppose that gravity is uniform, and points down. One can show that when the angle $\theta(t)$ remains close to 0 throughout the motion, so that we can make the *small angle approximation* $\sin(\theta) \approx \theta$, to a good approximation $\theta(t)$ satisfies the differential equation

$$mL^2 \frac{d^2\theta}{dt^2} + dL\frac{d\theta}{dt} + mgL\theta = 0,$$

where $d > 0$ is a damping constant, and g is the gravitational constant.

i) In terms of the constants m, L, d, and g, characterize when the pendulum is underdamped, critically damped, and overdamped. Briefly describe a typical motion of the pendulum in each of these three cases.

Suppose that $d/mL = 6$ and $g/L = 9$. In the absence of a driving torque, the motion of the pendulum is then described by the equation

$$\frac{d^2\theta}{dt^2} + 6\frac{d\theta}{dt} + 9\theta = 0. \tag{2}$$

ii) Find the roots of the characteristic polynomial $\chi(z)$ of equation (2).

iii) Find a basis $\{\phi_1, \phi_2\}$ of the space of solutions of equation (2).

iv) Compute the Wronskian $W(\phi_1, \phi_2)(t)$ of ϕ_1 and ϕ_2. Using the result, verify that ϕ_1 and ϕ_2 are linearly independent.

If a clockwise torque of magnitude $te^{-3t}mL^2$ is applied to the pendulum, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + 6\frac{d\theta}{dt} + 9\theta = te^{-3t}. \tag{3}$$

v) Solve equation (3) subject to the initial conditions

$$\theta(0) = 0, \quad \frac{d\theta}{dt}(0) = 0,$$

using your preferred method.