1. When a matrix A is diagonalizable, the corresponding diagonal matrix is the Jordan canonical form of A. Therefore, we already have a way of finding a basis that puts a matrix into its Jordan canonical form in the diagonalizable case — take the eigenvectors of the matrix as the basis.

The purpose of this problem is to develop a way of finding a basis of $\mathbb{R}^2$ in which a nondiagonalizable 2×2 matrix is in its Jordan canonical form.

Let A be a 2×2 matrix whose characteristic polynomial $p_A(z) = \det(A - zI) = (z - \lambda)^2$ has λ as a real double root.

Let

$$E_\lambda = \left\{ \mathbf{v} \in \mathbb{R}^2 : A\mathbf{v} = \lambda \mathbf{v} \right\}$$

be the eigenspace corresponding to λ.

Suppose that

$$\dim E_\lambda = 1.$$

Thus, λ is an eigenvalue of A with algebraic multiplicity 2 and geometric multiplicity 1.

The following theorem has many applications.

Thm (Cayley-Hamilton)**.** *Let $p_A(z)$ be the characteristic polynomial of A. Then $p_A(A) = 0$.*

In our case, $p_A(z) = (z - \lambda)^2$, so the Cayley-Hamilton theorem says that $(A - \lambda I)^2 = 0$. This last fact is useful in finding the Jordan basis.

To be self-contained, let's begin by showing that $(A - \lambda I)^2 = 0$ with an elementary computation, without use of the Cayley-Hamilton theorem. Label the entries of A as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

i) Show that

$$p_A(z) = z^2 - (a + d)z + (ad - bc).$$

The function $a + d$ of the entries of the matrix is called the *trace of* A and the function $ad - bc$ is called the *determinant of* A.

ii) Conclude from the hypothesis $p_A(z) = (z - \lambda)^2$ that

$$\begin{aligned} a + d &= 2\lambda \\ ad - bc &= \lambda^2. \end{aligned} \tag{1}$$

iii) Show that the two equalities of (1) imply that $\dfrac{(a-d)^2}{4} + bc = 0$.

iv) Show that $\lambda = (a + d)/2$ and, using the result of part iii), conclude that

$$(A - \lambda I)^2 = 0.$$

Now, let $\mathbf{w}$ be any nonzero vector in $\mathbb{R}^2$ but not in E_λ (such vectors exist, because $\mathbb{R}^2$ is two-dimensional and E_λ is one-dimensional). Let $\mathbf{v} = (A - \lambda I)\mathbf{w}$.

v) Show that $A\mathbf{w} = \mathbf{v} + \lambda\mathbf{w}$.

vi) Show that $(A - \lambda I)\mathbf{v} = 0$. Conclude that $A\mathbf{v} = \lambda\mathbf{v}$. Conclude that $\mathbf{v} \in E_\lambda$, and that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

vii) Conclude that the matrix of A with respect to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$ is

$$[A]_{\{\mathbf{v},\mathbf{w}\}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Therefore, A is in Jordan canonical form in the basis $\{\mathbf{v}, \mathbf{w}\}$!

viii) Find a basis that puts the matrix

$$B = \begin{pmatrix} 10 & 4 \\ -9 & -2 \end{pmatrix}.$$

into Jordan canonical form. (Find the eigenvalue of B and its eigenspace, then choose a nonzero $\mathbf{w}$ outside of the eigenspace, and take $\mathbf{v} = (B - \lambda I)\mathbf{w}$.)

ix) Show that

$$\exp(Bt) = \begin{pmatrix} e^{4t} + 6t\,e^{4t} & 4t\,e^{4t} \\ -9t\,e^{4t} & e^{4t} - 6t\,e^{4t} \end{pmatrix}.$$

x) Sketch the flow lines of the following first-order system, and find a parametrization of the flow line through the given point:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ -9 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 2,\ y(0) = -1.$$

The way one finds Jordan canonical forms of larger matrices has a similar flavour, building bases out of chains of vectors of the form $(A - \lambda I)^k\mathbf{w}$. A complete treatment, including a proof of existence of Jordan canonical form for an arbitrary matrix with complex entries, is done in Math 212. Alternatively, a good reference is the textbook Friedberg, Insel, Spence, *Linear Algebra*, §7.1–7.2.

2. For each of the following two systems, find the solution using the Method of Variation of Parameters.

i) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

ii) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$