1. For each of the following systems, sketch the flow lines of the system, and find a parametrization of the flow line that passes through the given point.

i) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 4,\ y(0) = 0.$

ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 1,\ y(0) = 0.$

iii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 0,\ y(0) = 2.$

2. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Show that $AB \neq BA$ and $\exp(A+B) \neq \exp(A)\exp(B)$.

3. Let $0 < \lambda_1 < \lambda_2$. Show that for the solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = x_0 \neq 0,\ y(0) = y_0,$$

the slopes of the tangent lines to the solution tend toward 0 as the solution approaches the origin (that is, as $t \to -\infty$).

(*Suggestion:* One way to approach this is to make use of the identity

$$\dot{y}(t_0)/\dot{x}(t_0) = \text{Slope of tangent line to } t \mapsto (x(t), y(t)) \text{ at } (x(t_0), y(t_0)).)$$

4. Sketch the flow lines of a few of the systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a \neq 1$$

as $a \to 1$ (for instance, with $a = 2/4$, $3/4$, $5/4$ and $6/4$), and compare them with the flow lines of the limiting system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What happens to the eigenvectors of the matrix $\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$ as $a \to 1$?