1. It is sometimes possible to find the curves traced out by solutions[1] of a first-order system of the form

$$\begin{aligned} \dot{x} &= v_1(x,y) \\ \dot{y} &= v_2(x,y) \end{aligned} \tag{1}$$

without finding the solutions of the system. The method is to make use of the identity

$$\frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}, \qquad \dot{x} \neq 0$$

that holds whenever the solution $t \mapsto (x(t), y(t))$ lies on the graph of a (differentiable) function $y(x)$ (as we have seen in the first lecture!). Thereby, we obtain the differential equation

$$\frac{dy}{dx} = \frac{v_2(x,y)}{v_1(x,y)} \tag{2}$$

that is satisfied by any function whose graph contains a solution of the system (1).

i) We have seen that the solutions of the system

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \end{aligned} \qquad \text{are} \qquad t \mapsto e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad t \in \mathbb{R},$$

where $x(0) = x_0,\ y(0) = y_0$ are the initial conditions. In other words, the solutions are paths that trace out open rays starting at the origin and going radially outward to infinity, as well as the equilibrium solution $t \mapsto (0,0)$.

Solve equation (2) for this system, and check that the flow lines (with the exception of the two vertical ones) lie on graphs of the solutions of (2).

ii) Similarly, we have seen that the solutions of the system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned} \qquad \text{are the circles} \qquad t \mapsto (A\cos(t+\phi),\, A\sin(t+\phi)), \quad t \in \mathbb{R},$$

where A and ϕ are determined by the initial conditions $x(0) = x_0,\ y(0) = y_0$ via $A = \sqrt{x_0^2 + y_0^2},\ \tan\phi = y_0/x_0$. Solve equation (2) for this system, and check that your findings are consistent our knowledge of the flow lines.

iii) Solve equation (2) for the following three systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned}, \qquad \begin{aligned} \dot{x} &= 1 + 2y \\ \dot{y} &= 1 + 3x^2 \end{aligned}, \qquad \begin{aligned} \dot{x} &= 2x^6y - 8x^4y^3 + 6x^2y^5 \\ \dot{y} &= 8x^3y^4 - 6x^5y^2 - 2xy^6 \end{aligned},$$

and sketch a few of the solutions (either graphs of solutions, or curves that are implicit solutions) — you can use a computer to help with the sketches (you are not asked to solve the three systems).

[1] More precisely, find curves containing the curves traced out by solutions.

2. i) Write down a first-order system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{3}$$

that is equivalent to the equation

$$\frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0, \qquad a_j \in \mathbb{R}. \tag{4}$$

ii) Let $p_A(z) = \det(A - zI)$ be the characteristic polynomial of the linear map (or matrix) A in (3). Show that

$$p_A(z) = (-1)^r \chi(z),$$

where $\chi(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_1 z + a_0$ is the characteristic polynomial of (4). Conclude that the roots of $\chi(z)$ are exactly the eigenvalues of A, with the same multiplicities.

Suggestion: Try the cases $r = 2, 3, 4$ first to help see how to do the computation in general.

3. The Laplace transform method applies just as well to systems of differential equations. Solve the first-order system

$$\begin{aligned} \dot{x} &= x - 5y \\ \dot{y} &= x - y \end{aligned}, \qquad x(0) = 1,\ y(0) = 0, \tag{5}$$

as follows:

i) Take Laplace transforms of both expressions, obtaining a system of (algebraic) equations in $\mathscr{L}[x](s)$ and $\mathscr{L}[y](s)$.

ii) Solve the system of equations from part i) for $\mathscr{L}[x](s)$ and $\mathscr{L}[y](s)$.

iii) Take inverse Laplace transforms to obtain the solution $t \mapsto (x(t), y(t))$ of the system (5).

4. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Compute the matrix exponential $\exp(A)$.