Mthe 237 — Problem Set 06 Solutions

1. Solve the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 2t, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1 \tag{1}$$

using the method of undetermined coefficients, as follows:

i) Find an annihilator $p_F\left(\frac{d}{dt}\right)$ of the function $F(t) = 2t$. That is, find a polynomial differential operator $p_F\left(\frac{d}{dt}\right)$ with constant coefficients such that

$$p_F\left(\frac{d}{dt}\right)(2t) = 0.$$

ii) Find a basis of solutions of the homogeneous linear differential equation

$$\left[p_F\left(\frac{d}{dt}\right)\left(\frac{d^2}{dt^2} + \frac{d}{dt} - 2\right)\right]y = 0.$$

iii) Find a particular solution ϕ_p of (1) in the span of the basis elements found in part ii).

(For this step, it helps save some work to disregard the basis elements that are solutions of the homogeneous linear equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$ associated to (1).)

iv) Find a solution of (1) satisfying the given initial conditions.

(As a reminder, we have shown in lecture that the set of solutions of (1) is equal to

$$\left\{\phi_p + \phi_h : \phi_h \text{ is a solution of the homogeneous linear equation } \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0 \text{ associated to (1)}\right\}.)$$

Solution. i) One can see that $p_F\left(\frac{d}{dt}\right) = \frac{d^2}{dt^2}$ works.

Alternatively, the function $F(t) = 2t$ is contained in the span of $\{1, t\}$, which is the basis of the differential equation whose characteristic polynomial has a double root at $z = 0$. The differential equation with characteristic polynomial $\chi(z) = z^2$ is $\frac{d^2y}{dt^2} = 0$, so we find again that we can take $p_F\left(\frac{d}{dt}\right) = \frac{d^2}{dt^2}$.

ii) The operator $\left(\frac{d^2}{dt^2} + \frac{d}{dt} - 2\right)$ factors as $\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 2\right)$, so we are looking for a basis of solutions of the differential equation

$$\frac{d^2}{dt^2}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 2\right)y = 0.$$

The characteristic polynomial of this differential equation is

$$\chi(z) = z^2(z-1)(z+2),$$

which has a double root at $z = 0$, and single roots at $z = 1$ and $z = -2$.

Therefore,

$$\{e^{0t}, te^{0t}, e^t, e^{-2t}\} = \{1, t, e^t, e^{-2t}\}$$

is a basis of solutions.

iii) Since e^t and e^{-2t} are solutions of the associated homogeneous equation

$$\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 2\right)y = 0,$$

we can disregard them for this step.

Therefore, we look for a particular solution of the form

$$\phi_p(t) = c_1 + c_2 t, \qquad c_1, c_2 \in \mathbb{R}.$$

Differentiating, we have

$$\begin{aligned}\phi_p'(t) &= c_2,\\ \phi_p''(t) &= 0.\end{aligned}$$

So that

$$\phi_p'' + \phi_p' - 2\phi_p = (0) + (c_2) - 2(c_1 + c_2 t) = (c_2 - 2c_1) + (-2c_2)t.$$

For ϕ_p to be a solution of (1), this should be equal to $F(t) = 2t$. By comparing coefficients, we obtain the system of linear equations

$$\begin{aligned}c_2 - 2c_1 &= 0,\\ -2c_2 &= 2.\end{aligned}$$

From the second equation, $c_2 = -1$, so that $c_1 = -1/2$.

The function

$$\phi_p(t) = -\frac{1}{2} - t$$

is a particular solution of (1).

iv) The affine space of all solutions is equal to

$$-\frac{1}{2} - t + b_1 e^t + b_2 e^{-2t}, \qquad b_1, b_2 \in \mathbb{R}.$$

The derivative of one of these functions is equal to

$$-1 + b_1 e^t - 2b_2 e^{-2t}.$$

Imposing the initial conditions, we arrive at the system of equations

$$\begin{aligned}-\frac{1}{2} - 0 + b_1 e^0 + b_2 e^0 &= 0\\ -1 + b_1 e^0 - 2b_2 e^0 &= 1,\end{aligned}$$

or

$$b_1 + b_2 = \frac{1}{2}$$
$$b_1 - 2b_2 = 2.$$

This system of equations has solution $b_1 = 1$, $b_2 = -\frac{1}{2}$, so the solution satisfying the initial conditions is

$$-\frac{1}{2} - t + e^t - \frac{1}{2}e^{-2t}.$$

2. i) Suppose that for $k = 1, \dots, n$, the function $\phi_k : I \to \mathbb{R}$ is a solution of the differential equation

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F_k(t).$$

Show that then the sum $\phi = \phi_1 + \cdots + \phi_n$ is a solution of the equation

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F_1(t) + \cdots + F_n(t).$$

ii) Suppose that for $k = 1, \dots, n$, the polynomial operator $p_{F_k}\left(\frac{d}{dt}\right)$ with constant coefficients is an annihilator of the function F_k over I. Show that then the product

$$p_{F_1}\left(\frac{d}{dt}\right) p_{F_2}\left(\frac{d}{dt}\right) \cdots p_{F_n}\left(\frac{d}{dt}\right)$$

is an annihilator of the function $F_1 + \cdots + F_n$ over I.

Solution. i) Let $p\left(\frac{d}{dt}\right)$ denote the differential operator

$$p\left(\frac{d}{dt}\right) = \frac{d^r}{dt^r} + a_{r-1}(t)\frac{d^{r-1}}{dt^{r-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t).$$

Because $p\left(\frac{d}{dt}\right)$ has *time-varying* coefficients, we must be careful not to assume that it has the same properties as differential operators with constant coefficients, such as commuting with other differential operators. However, it is clear that as a consequence of linearity of the derivative that $p\left(\frac{d}{dt}\right)$ is linear, meaning that

$$p\left(\frac{d}{dt}\right)(c_1\phi_1 + \cdots + c_n\phi_n) = c_1 p\left(\frac{d}{dt}\right)(\phi_1) + \cdots + c_n p\left(\frac{d}{dt}\right)(\phi_n) \qquad \text{for all } c_j \in \mathbb{R},\ \phi_j \in C^r(I, \mathbb{R}).$$

Then,

$$p\left(\frac{d}{dt}\right)(\phi) = p\left(\frac{d}{dt}\right)(\phi_1 + \cdots + \phi_n) = p\left(\frac{d}{dt}\right)(\phi_1) + \cdots + p\left(\frac{d}{dt}\right)(\phi_n) = F_1 + \cdots + F_n,$$

so that ϕ is indeed a solution of the claimed differential equation.

Here is a simpler way of writing the same solution:

We have

$$\begin{aligned}\frac{d^r\phi}{dt^r}+\cdots+a_0(t)(\phi) &= \frac{d^r(\phi_1+\cdots+\phi_n)}{dt^r}+\cdots+a_0(t)(\phi_1+\cdots+\phi_n)\\ &= \left(\frac{d^r\phi_1}{dt^r}+\cdots+a_0(t)\phi_1\right)+\cdots+\left(\frac{d^r\phi_n}{dt^r}+\cdots+a_0(t)\phi_n\right)\\ &= F_1+\cdots+F_n.\end{aligned}$$

ii) Applying $p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)$ to $F_1+\cdots+F_n$, we have

$$\begin{aligned}&\left(p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)\right)(F_1+\cdots+F_n)\\ &=\left(p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)\right)(F_1)+\cdots+\left(p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)\right)(F_n).\end{aligned}$$

Because polynomial differential operators with constant coefficients commute, the j-th summand in the last expression is equal to

$$\left(p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)\right)(F_j)=\left(\prod_{k\neq j}p_{F_j}\left(\frac{d}{dt}\right)\right)\left[p_{F_j}\left(\frac{d}{dt}\right)(F_j)\right]=0,$$

where $\prod_{k\neq j}p_{F_j}\left(\frac{d}{dt}\right)$ denotes the product over all indices $k\neq j$.

Since every summand is equal to 0, the sum is equal to 0, which shows that $p_{F_1}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)$ annihilates $F_1+\cdots+F_n$.

3. Solve the differential equation

$$\frac{d^2y}{dt^2}+16y=t^2+2\cos(2t)\sin(2t),\qquad y(0)=\frac{127}{128},\quad \frac{dy}{dt}(0)=\frac{7}{8}$$

using the method of undetermined coefficients.
(Using the sine angle addition identity, we have $\sin(2\theta)=2\sin(\theta)\cos(\theta)$. To handle the right-hand side, it may help to apply either of the results of Problem 2.)

Solution. Using the sine angle addition identity, we recognize the right side as a quasipolynomial. The differential equation we are to solve is

$$\frac{d^2y}{dt^2}+16y=t^2+\sin(4t). \tag{2}$$

Let $F_1(t)=t^2$ and $F_2(t)=\sin(4t)$.

We see that F_1 is contained in the span of $\{1, t, t^2\}$, which is a basis for the space of solutions of $\frac{d^3y}{dt^3}=0$, so we may take $p_{F_1}\left(\frac{d}{dt}\right)=\frac{d^3}{dt^3}$.

The function F_2 is contained in the span of $\{\cos(4t), \sin(4t)\}$, which is a basis for the space of solutions of $\left(\frac{d}{dt}-4i\right)\left(\frac{d}{dt}+4i\right)y=0$, so we may take $p_{F_2}\left(\frac{d}{dt}\right)=\left(\frac{d}{dt}-4i\right)\left(\frac{d}{dt}+4i\right)$.

By Problem 2, part ii), the operator product

$$\frac{d^3}{dt^3}\left(\frac{d}{dt}-4i\right)\left(\frac{d}{dt}+4i\right)$$

is then an annihilator of $t^2+\sin(4t)$.

Applying this annihilator to both sides of (2), we obtain the homogeneous equation

$$\frac{d^3}{dt^3}\left(\frac{d}{dt}-4i\right)\left(\frac{d}{dt}+4i\right)\left(\frac{d^2}{dt^2}+16\right)y=0.$$

Now, $\left(\frac{d^2}{dt^2}+16\right)=\left(\frac{d}{dt}-4i\right)\left(\frac{d}{dt}+4i\right)$. So the above equation is

$$\frac{d^3}{dt^3}\left(\frac{d}{dt}-4i\right)^2\left(\frac{d}{dt}+4i\right)^2 y=0.$$

The basis of solutions of this equation is

$$\left\{1,\, t,\, t^2,\, \cos(4t),\, t\cos(4t),\, \sin(4t),\, t\sin(4t)\right\}.$$

Discarding the solutions $\cos(4t)$, $\sin(4t)$ of the homogeneous equation associated to (2) for the moment, we look for a particular solution of the form

$$\phi_p(t)=c_1+c_2t+c_3t^2+c_4\,t\cos(4t)+c_5\,t\sin(4t).$$

Differentiating, we find

$$\begin{aligned}
\phi_p'(t)&=c_2+2c_3t+c_4\left(\cos(4t)-4t\sin(4t)\right)+c_5\left(\sin(4t)+4t\cos(4t)\right)\\
&=c_2+2c_3t+c_4\cos(4t)+c_5\sin(4t)+4c_5\,t\cos(4t)-4c_4\,t\sin(4t),\\
\phi_p''(t)&=2c_3-4c_4\sin(4t)+4c_5\cos(4t)+4c_5\left(\cos(4t)-4t\sin(4t)\right)-4c_4\left(\sin(4t)+4t\cos(4t)\right)\\
&=2c_3+8c_5\cos(4t)-8c_4\sin(4t)-16c_4\,t\cos(4t)-16c_5\,t\sin(4t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\phi_p''+16\phi_p\\
&=2c_3+8c_5\cos(4t)-8c_4\sin(4t)-16c_4\,t\cos(4t)-16c_5\,t\sin(4t)+16(c_1+c_2t+c_3t^2+c_4\,t\cos(4t)+c_5\,t\sin(4t))\\
&=(16c_1+2c_3)+16c_2\,t+16c_3\,t^2+8c_5\,\cos(4t)-8c_4\,\sin(4t)+(16c_4-16c_4)\,t\cos(4t)+(16c_5-16c_5)\,t\sin(4t)\\
&=(16c_1+2c_3)+16c_2\,t+16c_3\,t^2+8c_5\,\cos(4t)-8c_4\,\sin(4t).
\end{aligned}$$

Comparing coefficients with $t^2+\sin(4t)$, we find the system of equations

$$\begin{aligned}
16c_1+2c_3&=0\\
16c_2&=0\\
16c_3&=1\\
8c_5&=0\\
-8c_4&=1.
\end{aligned}$$

This system has solution

$$c_1 = -\frac{1}{128},\ c_2 = 0,\ c_3 = \frac{1}{16},\ c_4 = -\frac{1}{8},\ c_5 = 0.$$

Therefore, we found the particular solution

$$\phi_p(t) = -\frac{1}{128} + \frac{t^2}{16} - \frac{t\cos(4t)}{8}$$

of equation (2). The affine space of all solutions of (2) is

$$-\frac{1}{128} + \frac{t^2}{16} - \frac{t\cos(4t)}{8} + b_1\cos(4t) + b_2\sin(4t), \quad b_1,\ b_2 \in \mathbb{R}.$$

The derivative of such a solution is

$$\frac{t}{8} - \frac{\cos(4t) - 4t\sin(4t)}{8} - 4b_1\sin(4t) + 4b_2\cos(4t).$$

Imposing the initial conditions, we get the system of equations

$$\begin{aligned} -\frac{1}{128} + 0 - 0 + b_1 + 0 &= \frac{127}{128}, \\ 0 - \frac{1-0}{8} - 0 + 4b_2 &= \frac{7}{8}, \end{aligned}$$

which has the solution

$$b_1 = 1,\ b_2 = \frac{1}{4}.$$

So that the solution satisfying the initial conditions is

$$-\frac{1}{128} + \frac{t^2}{16} - \frac{t\cos(4t)}{8} + \cos(4t) + \frac{\sin(4t)}{4}.$$