1. Solve the following differential equations.

i) $\dfrac{dy}{dt} + 2017y = 0, \quad y(0) = 5.$

ii) $\dfrac{d^2y}{dt^2} - 10\dfrac{dy}{dt} + 21y = 0, \quad y(0) = 5, \quad \dfrac{dy}{dt}(0) = 19.$

iii) $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 25y = 0, \quad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 15.$

iv) $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 16y = 0, \quad y(0) = 1, \quad \dfrac{dy}{dt}(1) = 10.$

v) $\dfrac{d^6y}{dt^6} - y = 0.$ (It is sufficient to find a basis for the space of solutions.)

2. In each of the following, find a linear homogeneous differential equation with constant coefficients with the given functions as a basis for its space of solutions.

i) $\phi_1(t) = e^t, \quad \phi_2(t) = e^{2t}.$

ii) $\phi_1(t) = e^t, \quad \phi_2(t) = e^{2t}, \quad \phi_3(t) = e^{3t}.$

iii) $\phi_1(t) = e^{-kt}, \quad \phi_2(t) = t\,e^{-kt}, \quad \phi_3(t) = e^{2t},$ where k is a real number.

iv) $\phi_1(t) = e^{\sigma t}\cos(\omega t), \quad \phi_2(t) = e^{\sigma t}\sin(\omega t),$ where σ and $\omega \neq 0$ are real numbers.

v) $\phi_1(t) = 1, \quad \phi_2(t) = t, \quad \phi_3(t) = t^2, \quad \phi_4(t) = e^{-t}\cos(t), \quad \phi_5(t) = e^{-t}\sin(t).$

3. By comparing the real and imaginary parts of the identity

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi},$$

show the angle addition identities for sin and cos:

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi), \quad \text{and}$$
$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$$

4 (Simple harmonic motion)**.** Consider a mass hanging on a spring with spring constant k. After an initial stretch of the spring to balance the force of gravity, the mass will hang at rest. Choose a coordinate system such that the y-axis is aligned with the spring, and such that the rest point of the mass is at $y = 0$.

If the mass is moved a distance y from $y = 0$, it will be acted on by a restoring force due do the spring, given by Hooke's law: $\mathbf{F}_{\text{restoring}} = -ky$. In the absence of other forces (such as damping), the motion of the mass is described by

$$m\frac{d^2y}{dt^2} = -ky \qquad \text{(Newton's second law)},$$

or, bringing to standard form for a linear equation,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad \text{where } \omega^2 = k/m. \tag{1}$$

i) Find the roots of the characteristic polynomial of (1), and conclude that $\phi_1(t) = \cos(\omega t)$ and $\phi_2(t) = \sin(\omega t)$ are a basis for the space of solutions of (1).

ii) Check that for real numbers $A \geq 0$ and $\phi \in (-\pi, \pi]$, the function

$$A\cos(\omega t + \phi),$$

is a solution of (1). Therefore, we have

$$A\cos(\omega t + \phi) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

for some real numbers c_1, c_2. Find c_1 and c_2 in terms of A and ϕ. (*Suggestion:* Expand $\cos(\omega t + \phi)$ using the angle addition identity, and make use of the fact that any vector is expressed uniquely as a linear combination of basis elements.)

iii) Find A and ϕ so that the function $A\cos(\omega t+\phi)$ is a solution of (1) with initial conditions

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0\omega.$$

Conclude that any solution of (1) may be written in the form $A\cos(\omega t + \phi)$.

iv) The parameters ω, A and ϕ are called the frequency, amplitude and phase of the motion, respectively. Briefly describe (with sketches, if possible) how the graph of $A\cos(\omega t+\phi)$ depends on these three parameters.