1. i) Check that the differential equation $(y^2+y)-x\frac{dy}{dx}=0$ is not exact. Now, multiplying both sides by $\mu(x,y)=y^{-2}$ yields the differential equation $\left(1+\frac{1}{y}\right)-\frac{x}{y^2}\frac{dy}{dx}=0$. Check that the latter is exact (for $y\neq 0$).

If a differential equation that is not exact is converted to one that is exact by multiplying both sides of the equation by a function $\mu(x,y)$, as in the example of i), the function $\mu(x,y)$ is called an *integrating factor.*

Solutions of the equation multiplied by μ are also solutions of the original equation, as long as we avoid regions where μ is zero.

Although in principle an integrating factor exists for every equation of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0, \tag{1}$$

in practice an integrating factor is frequently hard to find, and there is no known general method for finding it.

One approach to trying to find an integrating factor is to guess that it has a special simple form, and try your luck. For example, suppose we *guessed* that the differential equation (1) has an integrating factor $\mu(x)$ that is a function of x only, so that

$$\mu(x)M(x,y)+\mu(x)N(x,y)\frac{dy}{dx}=0$$

is now exact.

ii) Show that the exactness condition

$$\frac{\partial(\mu(x)M(x,y))}{\partial y}=\frac{\partial(\mu(x)N(x,y))}{\partial x}$$

may be rearranged to

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}. \tag{2}$$

If the right side of eq. (2) is a function of x only, then eq. (2) is separated equation for $\mu(x)$, which we can solve by the usual method for separated equations (integrating both sides).

iii) The equation $(e^x-\sin(y))+\cos(y)\frac{dy}{dx}=0$ is not exact, but has an integrating factor that may be found by the above method. Find it.

iv) Check that for the equation $xy^2+x\frac{dy}{dx}=0$, the expression $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is not a function of x only. Therefore, the above method does not yield an integrating factor.

Other common guesses for the form of an integrating factor are $\mu(y)$, $\mu(xy)$, $\mu(x/y)$, $\mu(y/x)$. Each, in lucky cases, leads to a separable differential equation for μ.[1] However, all five may fail to yield an integrating factor. Because there is no known general method of finding integrating factors, their usefulness is limited.

The idea of integrating factors does lead to a general method of solving first-order linear equations, however, which we shall cover in detail in lecture!

[1] For instance, $\mu(xy)=1/(xy(1-xy))$ works for the equation of part iv).

2. i) Check that

$$W(e^t, te^t, t^2e^t)(t) = \det\begin{pmatrix} e^t & te^t & t^2e^t \\ (e^t)' & (te^t)' & (t^2e^t)' \\ (e^t)'' & (te^t)'' & (t^2e^t)'' \end{pmatrix} = 2e^{3t}, \quad t \in \mathbb{R},$$

and conclude that $\{e^t, te^t, t^2e^t\}$ is a linearly independent subset of $C^\infty(\mathbb{R}, \mathbb{R})$.

ii) Show that if

$$\alpha_1 e^t + \alpha_2 te^t + \alpha_3 t^2e^t = 0$$

for all $t \in \mathbb{R}$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (give a direct argument avoiding the use of Wronskians). This gives another argument that $\{e^t, te^t, t^2e^t\}$ is a linearly independent subset of $C^\infty(\mathbb{R}, \mathbb{R})$.

3. Let V and W be finite-dimensional vector spaces. Show that if there exists an isomorphism $L\colon V \to W$, then $\dim V = \dim W$.

(This fact was needed in the proof that the dimension of the space of solutions of a homogeneous linear equation is equal to its order. It is a particular case of the principle that isomorphic vector spaces have identical linear-algebraic properties.)

Suggestion: Let $\dim V = r$. Choose a basis $\{e_1, \ldots, e_r\}$ of V. Show that $\{L(e_1), \ldots, L(e_r)\}$ is a basis of W.

Optional Problem. Let I be the open interval $(-1, 1)$. Can you find a function in $C^0(I, \mathbb{R})$ but not $C^1(I, \mathbb{R})$? In $C^1(I, \mathbb{R})$ but not $C^2(I, \mathbb{R})$? In $C^r(I, \mathbb{R})$ but not $C^{r+1}(I, \mathbb{R})$?