1. Solve the following differential equations.
(Hand-in only the starred problems, but please attempt them all. Solutions can be left in implicit form. The integral $\int \frac{dx}{1+x^2} = \arctan(x)$ may be useful. There are suggestions at the bottom of p. 2, but try solving without looking at the suggestions first.)

i) $(3x^2y + 8xy^2) + (x^3 + 8x^2y + 12y^2)\frac{dy}{dx} = 0, \quad y(1) = 0.$

ii)* $\frac{dy}{dx} = x^2 + 2xy + y^2, \quad y(0) = 0.$

iii)* $(x-y)\frac{dy}{dx} = (x+y), \quad y(1) = 0.$

iv) $(x-y-1)\frac{dy}{dx} = (x+y+1), \quad y(1) = -1.$

v) $\frac{dy}{dx} = e^{x+y}, \quad y(0) = 0.$

vi) $\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dx} = 0, \quad y(1) = 0.$

vii)* $(x - 2xy + e^y) + (y - x^2 + xe^y)\frac{dy}{dx} = 0, \quad y(0) = 1.$

viii) $x\frac{dy}{dx} = xe^{y/x} + y, \quad y(1) = 0.$

2. For each of the following differential equations, is existence of a solution in some nonempty open interval about x_0 implied by the Existence and Uniqueness Theorem for First-Order Ordinary Differential Equations? If so, is uniqueness?

i) $\frac{dy}{dx} = x^2 + 2xy + y^2, \quad y(0) = 0.$

ii) $\frac{dy}{dx} = \frac{1}{x^2+y^2+1}, \quad y(0) = 1.$

iii) $\frac{dy}{dx} = y^{1/5}, \quad y(1) = 1.$

iv) $\frac{dy}{dx} = y^{1/5}, \quad y(2) = 0.$

3. In this problem, we look at water flowing out of a container through an opening at its bottom.

Let $h(t)$ denote the height of the water level above the bottom of the container at time t, let $A(t)$ denote the area of the top surface of the water at time t, and let a denote the cross-sectional area of the opening at the bottom. It follows from conservation of energy (Torricelli's principle) that

$$A(t)\,\frac{dh}{dt}(t) = -a\sqrt{2g\,h(t)}, \tag{1}$$

where g is the gravitational constant.

Consider three possible containers—

i) Solve eq. (1) for a cylindrical container of height h_0 with cross-sectional area A (so that $A(t) = A$ is a constant function) that is standing upright (that is, on the circular face that has the opening). The initial condition is that the container is full at $t = 0$, so that $h(0) = h_0$.

ii) Now consider a circular cone of height h_0 and with diameter $2h_0$ across the top, standing upright on its vertex. Using similar triangles, argue that $A(t) = \pi h(t)^2$. Then, solve eq. (1) with initial condition $h(0) = h_0$.

iii) A paraboloid of revolution is the surface obtained by rotating the graph of the function $y(x) = x^2$ about the y-axis. For a container of height h_0 in the shape of a paraboloid of revolution, find a relationship between $A(t)$ and $h(t)$, and solve eq. (1). The initial condition is once again $h(0) = h_0$.

iv) In terms of h_0, how much time does it take for each of the three containers to completely empty out?

v) Supposing $a\sqrt{2g} = 1$ and $h_0 = 1$, sketch a plot of $h(t)$ versus t for each of the three containers (you may need to use a plotting program for this).

Optional Problem. The purpose of this question is to prove that the differential equation $dy/dx = x + y$ is not separable. (As a reminder, a differential equation is separable if it can be written in the form $n(y)\,dy/dx = m(x)$, or, equivalently, in the form $dy/dx = m(x)/n(y)$.)

i) Let $F(x, y)$ be a real-valued function of two variables. Show that if $F(x, y) = m(x)/n(y)$ for some functions m, n of a single variable, then for any real numbers x_0, x_1, y_0, y_1 such that the four points (x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_1) are in the domain of F,

$$F(x_0, y_0)F(x_1, y_1) - F(x_0, y_1)F(x_1, y_0) = 0.$$

ii) Conclude that $F(x, y) = x + y$ cannot be written in the form $F(x, y) = m(x)/n(y)$.

Suggestions for ***1.*** i) Exact; ii) Factor and try substitution $v = x + y$; iii) Homogeneous; iv) Substitute to reduce to iii; v) Separable; vi); Exact; vii) Exact; viii) Homogeneous.