1. For each of the following, find a function $y(x)$ that satisfies the differential equation and initial condition. Take care to provide the domain of definition of the solution.

i) $\dfrac{dy}{dx} = 5y^2, \quad y(0) = 1;$

ii) $x\dfrac{dy}{dx} = 2(y-9), \quad y(1) = 10;$

iii) $(x^2+1)\dfrac{dy}{dx} = xy, \quad y(0) = 1;$

iv) $\dfrac{dy}{dx} = xy + 2x + y + 2, \quad y(0) = -1;$

v) $2xy\dfrac{dy}{dx} = x^2 + y^2, \quad y(1) = \sqrt{3}$ (Corrected Sept. 20).

2. As we saw in lecture, if the path of a particle described parametrically by $(x(t), y(t))$, $t \in I$ lies on the graph of a function $y(x)$, then by the chain rule

$$\frac{dy}{dt}(t) = \frac{dy}{dx}(x(t))\frac{dx}{dt}(t),$$

so that

$$\frac{dy}{dx}(x(t)) = \frac{\frac{dy}{dt}(t)}{\frac{dx}{dt}(t)} \qquad \text{or, written more compactly,} \qquad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{whenever } \dot{x} \neq 0. \tag{1}$$

i) The path $t \mapsto (\sqrt{8}\cos(t), \sqrt{2}\sin(t))$, $t \in [0, 2\pi)$ describes an ellipse E. Using (1), find the slope of the tangent line to E at the point $(2,1)$.

As a reminder, the vector $(\dot{x}(t), \dot{y}(t))$ is called the velocity of the particle at time t.

Suppose that the velocity of a particle is perpendicular to its position for all $t \in I$, and that the path of the particle lies on the graph of the function $y(x)$. Under these hypotheses:

ii) Show that $x(t)\dot{x}(t) + y(t)\dot{y}(t) = 0$ for all $t \in I$.

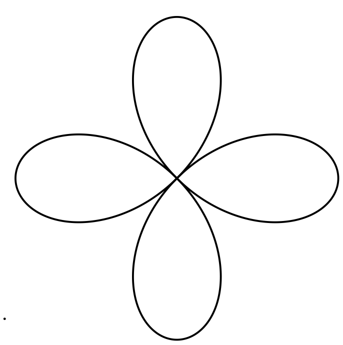
iii) Show that $y(x)$ is a solution to the differential equation

$$y\,\frac{dy}{dx} = -x.$$

iv) Conclude that the path of the particle lies on a circle.

Optional Problem. For those who enjoy playing around with (algebraic) equations (and can spare a bit of time): Let C be the curve in $\mathbb{R}^2$ defined implicitly by the equation

$$(x^2+y^2)^3 = (x^2-y^2)^2 \quad \text{(the quadrifolium, or four-leafed clover).}$$



Find the points of C where the hypotheses of the Implicit Function Theorem do not hold.

Answer: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{\frac{2}{27}}, \sqrt{\frac{10}{27}})$, $(\sqrt{\frac{2}{27}}, -\sqrt{\frac{10}{27}})$, $(-\sqrt{\frac{2}{27}}, \sqrt{\frac{10}{27}})$, $(-\sqrt{\frac{2}{27}}, -\sqrt{\frac{10}{27}})$.