Mthe 237 Final Exam

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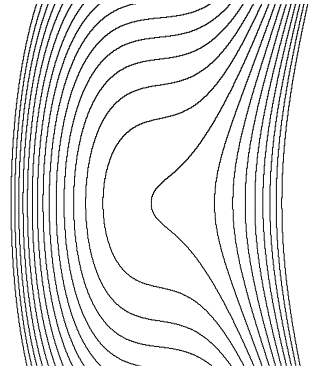


Instructions: This exam is three hours in duration.

Please write your answers in the booklets provided. Hand in both the booklets and the question paper.

To receive full credit, you must justify your answers. Answers with little or no justification will receive little or no credit.

Calculators, data sheets, notes, and other aids are not permitted. A table of Laplace transforms is provided on the last two pages.

Good Luck!

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Student Number: ____________________

1	2	3	4	5	Total

$$\textit{Common notations for derivatives: } \frac{dy}{dt} = \dot{y} = y'; \ \frac{d^2y}{dt^2} = \ddot{y} = y''; \ \frac{d^r y}{dt^r} = y^{(r)}.$$

1 (20 points)**.** Solve the following differential equations. You are not required to find the domain of definition of the solutions. Solutions may be given in implicit form.

i) $\dfrac{dy}{dt} = ay, \quad a \in \mathbb{R}, \qquad y(t_0) = y_0.$

ii) $\dfrac{dy}{dx} = x^2 - 2xy + y^2, \qquad y(0) = 0.$

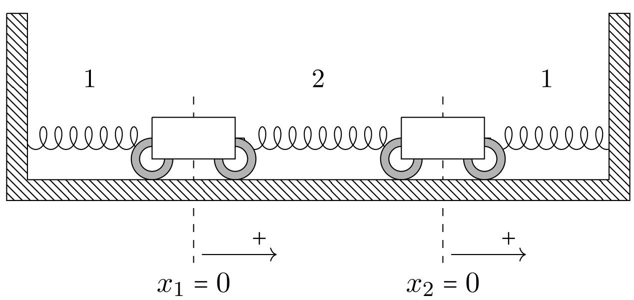
iii) $x\dfrac{dy}{dx} = xe^{y/x} + y, \qquad y(1) = 0.$

(*Hints:* For ii): Substitute $v = x - y$. One of the integrals will require a simple decomposition into partial fractions. For iii): Substitute $v = y/x$.)

2 (20 points)**.** Find all solutions of the differential equation

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = \cos(t).$$

3 (20 points)**.** Let x_1 denote the displacement of the left block in the diagram below from its rest position. Let x_2 denote the displacement of the right block from its rest position. The displacements x_1 and x_2 are taken to be positive toward the right.



Suppose that each spring has the spring constant given by the number above that spring in the diagram. Suppose that the mass of each block is equal to 1. Neglect friction.

In class, we have shown that x_1 and x_2 satisfy the system of equations

$$\begin{aligned} \ddot{x}_1 &= -3x_1 + x_2 \\ \ddot{x}_2 &= x_1 - 3x_2. \end{aligned}$$

Using the Laplace transform, find solutions $x_1(t)$ and $x_2(t)$ of the system subject to the initial conditions

$$x_1(0) = -1, \ \dot{x}_1(0) = 0, \quad \text{and} \quad x_2(0) = 1, \ \dot{x}_2(0) = 0.$$

Briefly describe the motion of the two blocks.

(*Hint:* $\mathscr{L}[x_1](s)$ and $\mathscr{L}[x_2](s)$ are rational functions with a degree one polynomial in the numerator and a degree two polynomial in the denominator.)

4 (25 points)**.** i) Sketch the solutions of the following first-order system, and find a parametrization of the solution passing through the given point:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

ii) Solve the first-order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 25 \\ -25 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

5 (15 points)**.** Find the parametrized path $\mathbf{x}$ in $\mathbb{R}^2$ with the following two properties:

1. $\mathbf{x}(0) = (1, 0)$,
2. The velocity vector of $\mathbf{x}$ makes an angle of $\pi/4$ radians clockwise with the position vector of $\mathbf{x}$, and is equal in magnitude to the position vector of $\mathbf{x}$, at all points of the path.

Reminder: Let $\mathbf{x}\colon t \mapsto (x(t),\, y(t)),\; t \in I$ be a parametrized path in $\mathbb{R}^2$. The *velocity* of $\mathbf{x}$ is the function $\dot{\mathbf{x}}\colon t \mapsto (\dot{x}(t),\, \dot{y}(t)),\; t \in I$.

Table of Laplace Transforms

Notation: $\mathbb{N} = \{0, 1, 2, \dots\}$ natural numbers, $\mathbb{R}$ real numbers.

$$u_a(t) = u_0(t-a) = \begin{cases} 1 & t \geq a \\ 0 & \text{otherwise} \end{cases}, \quad \text{unit step function with jump at } t = a.$$

1 General properties

$f(t),\ t \geq 0$	$\mathscr{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
$(f+g)(t)$	$\mathscr{L}[f](s) + \mathscr{L}[g](s)$
$(cf)(t),\ c \in \mathbb{R}$	$c\mathscr{L}[f](s)$
$\frac{df}{dt}(t)$	$s\mathscr{L}[f](s) - f(0)$
$\frac{d^2 f}{dt^2}(t)$	$s^2\mathscr{L}[f](s) - sf(0) - \frac{df}{dt}(0)$
$\frac{d^r f}{dt^r}(t),\ r \in \mathbb{N}$	$s^r\mathscr{L}[f](s) - s^{r-1}f(0) - s^{r-2}\frac{df}{dt}(0) - \cdots - \frac{d^{r-1}f}{dt^{r-1}}(0)$
$u_a(t)f(t-a),\ a \geq 0,$	$e^{-as}\mathscr{L}[f](s)$
$e^{\sigma s}f(t),\ \sigma \in \mathbb{R}$	$\mathscr{L}[f](s-\sigma)$, whereever converges
$(f * g)(t) = \int_0^t f(u)g(t-u)\,du$	$\mathscr{L}[f](s)\,\mathscr{L}[g](s)$

2 Transforms of Quasipolynomials and the Unit Step Functions

1	$\frac{1}{s}, \quad s > 0$
t	$\frac{1}{s^2}, \quad s > 0$
$t^k, \ k \in \mathbb{N}$	$\frac{k!}{s^{k+1}}, \quad s > 0$
$e^{\sigma t}, \ \sigma \in \mathbb{R}$	$\frac{1}{s-\sigma}, \quad s > \sigma$
$t\, e^{\sigma t}, \ \sigma \in \mathbb{R}$	$\frac{1}{(s-\sigma)^2}, \quad s > \sigma$
$t^k\, e^{\sigma t}, \ \sigma \in \mathbb{R}, k \in \mathbb{N}$	$\frac{k!}{(s-\sigma)^{k+1}}, \quad s > \sigma$
$\cos(\omega t), \ \omega \in \mathbb{R}$	$\frac{s}{s^2+\omega^2}, \quad s > 0$
$\sin(\omega t), \ \omega \in \mathbb{R}$	$\frac{\omega}{s^2+\omega^2}, \quad s > 0$
$t\, \cos(\omega t), \ \omega \in \mathbb{R}$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \quad s > 0$
$t\, \sin(\omega t), \ \omega \in \mathbb{R}$	$\frac{2\omega s}{(s^2+\omega^2)^2}, \quad s > 0$
$t^k \cos(\omega t), \ \omega \in \mathbb{R}, k \in \mathbb{N}$	$(-1)^k \frac{d^k}{ds^k}\left(\frac{s}{s^2+\omega^2}\right), \quad s > 0$
$t^k \sin(\omega t), \ \omega \in \mathbb{R}, k \in \mathbb{N}$	$(-1)^k \frac{d^k}{ds^k}\left(\frac{\omega}{s^2+\omega^2}\right), \quad s > 0$
$e^{\sigma t} \cos(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{s-\sigma}{(s-\sigma)^2+\omega^2}, \quad s > \sigma$
$e^{\sigma t} \sin(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{\omega}{(s-\sigma)^2+\omega^2}, \quad s > \sigma$
$t\, e^{\sigma t} \cos(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{(s-\sigma)^2-\omega^2}{((s-\sigma)^2+\omega^2)^2}, \quad s > \sigma$
$t\, e^{\sigma t} \sin(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{2(s-\sigma)\omega}{((s-\sigma)^2+\omega^2)^2}, \quad s > \sigma$
$u_a(t), \ a \geq 0$	$\frac{e^{-as}}{s}, \quad s > 0$