Mthe 237 Supplemental Final Exam

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Queen's University Applied Science
Department of Mathematics and Statistics
Instructor: Ilia Smirnov

Instructions: This exam is three hours in duration.

Please write your answers in the booklets provided. Hand in both the booklets and the question paper.

To receive full credit, you must justify your answers. Answers with little or no justification will receive little or no credit.

Calculators, data sheets, notes, and other aids are not permitted. A table of Laplace transforms is provided on the last two pages.

Good Luck!

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Student Number: ____________________

1	2	3	4	Total

Common notations for derivatives: $\frac{dy}{dt} = \dot{y} = y'$; $\frac{d^2y}{dt^2} = \ddot{y} = y''$; $\cdots$; $\frac{d^r y}{dt^r} = y^{(r)}$.

1 (30 points)**.** Solve the differential equation

$$\frac{d^2y}{dt^2} + y = \cos(t), \quad y(0) = 1, \ \frac{dy}{dt}(0) = 0$$

a) using the Method of Undetermined Coefficients (a.k.a. Annihilator Method);

b) using the Laplace Transform Method;

c) using Variation of Parameters.
[*Possibly useful identities:* $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1, \quad \sin(2t) = 2\sin(t)\cos(t)$.]

2 (20 points)**.** A rocket is propelled by burning fuel and expelling it backwards (from the point of view of an observer on the rocket). As the burned fuel gets expelled, the mass of the rocket decreases with time.

Assuming that the only other force on the rocket is gravity (and the rocket is sufficiently close to the surface of the Earth), the motion of the rocket (that is taking off vertically) satisfies the following modified version of Newton's second law (that takes into account the variation of mass with time):

$$m\ddot{y} = u\dot{m} - mg,$$

where the variables denote the following

$y = y(t)$	Vertical position of the rocket
$m = m(t)$	Mass of the rocket
$\dot{m} = \frac{dm}{dt}$	Rate of change of mass with time
$u = u(t)$	Velocity of the expelled mass relative to an observer on the rocket
g	Gravitational constant

Let M be the dry mass of the rocket (i.e. mass without any fuel), m_0 be the initial mass of the fuel, and suppose that the fuel is burned at a constant rate k and expelled backwards relative to the rocket at a constant velocity U. The mass of the rocket at time t is then given by $m(t) = M + m_0 - kt$.

Suppose that the rocket starts at rest at $y = 0$, so that $y(0) = 0$ and $\dot{y}(0) = 0$.

a) Let $a > 0$ be a constant. Verify that

$$\int \log(1 - at)\, dt = \left(t - \frac{1}{a}\right)\log(1 - at) - \left(t - \frac{1}{a}\right).$$

b) Find the velocity and position of the rocket as functions of time from $t = 0$ until the time when the rocket runs out of fuel.

3 (20 points). a) Define $\exp(A)$, where A is an $n \times n$ matrix.

b) Compute $\exp(At)$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4 (30 points). i) Sketch the solutions of the following first-order system, and find a parametrization of the solution passing through the given point:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

ii) Solve the first-order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 25 \\ -25 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Table of Laplace Transforms

Notation: $\mathbb{N} = \{0, 1, 2, \ldots\}$ natural numbers, $\mathbb{R}$ real numbers.

$$u_a(t) = u_0(t-a) = \begin{cases} 1 & t \geq a \\ 0 & \text{otherwise} \end{cases}, \quad \text{unit step function with jump at } t = a.$$

1 General properties

$f(t),\ t \geq 0$	$\mathscr{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
$(f+g)(t)$	$\mathscr{L}[f](s) + \mathscr{L}[g](s)$
$(cf)(t),\ c \in \mathbb{R}$	$c\mathscr{L}[f](s)$
$\frac{df}{dt}(t)$	$s\mathscr{L}[f](s) - f(0)$
$\frac{d^2f}{dt^2}(t)$	$s^2\mathscr{L}[f](s) - sf(0) - \frac{df}{dt}(0)$
$\frac{d^rf}{dt^r}(t),\ r \in \mathbb{N}$	$s^r\mathscr{L}[f](s) - s^{r-1}f(0) - s^{r-2}\frac{df}{dt}(0) - \cdots - \frac{d^{r-1}f}{dt^{r-1}}(0)$
$u_a(t)f(t-a),\ a \geq 0,$	$e^{-as}\mathscr{L}[f](s)$
$e^{\sigma s}f(t),\ \sigma \in \mathbb{R}$	$\mathscr{L}[f](s-\sigma)$, whereever converges
$(f * g)(t) = \int_0^t f(u)g(t-u)\,du$	$\mathscr{L}[f](s)\,\mathscr{L}[g](s)$

2 Transforms of Quasipolynomials and the Unit Step Functions

1	$\frac{1}{s}, \quad s > 0$
t	$\frac{1}{s^2}, \quad s > 0$
$t^k, \ k \in \mathbb{N}$	$\frac{k!}{s^{k+1}}, \quad s > 0$
$e^{\sigma t}, \ \sigma \in \mathbb{R}$	$\frac{1}{s-\sigma}, \quad s > \sigma$
$t\, e^{\sigma t}, \ \sigma \in \mathbb{R}$	$\frac{1}{(s-\sigma)^2}, \quad s > \sigma$
$t^k\, e^{\sigma t}, \ \sigma \in \mathbb{R}, k \in \mathbb{N}$	$\frac{k!}{(s-\sigma)^{k+1}}, \quad s > \sigma$
$\cos(\omega t), \ \omega \in \mathbb{R}$	$\frac{s}{s^2+\omega^2}, \quad s > 0$
$\sin(\omega t), \ \omega \in \mathbb{R}$	$\frac{\omega}{s^2+\omega^2}, \quad s > 0$
$t\cos(\omega t), \ \omega \in \mathbb{R}$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \quad s > 0$
$t\sin(\omega t), \ \omega \in \mathbb{R}$	$\frac{2\omega s}{(s^2+\omega^2)^2}, \quad s > 0$
$t^k\cos(\omega t), \ \omega \in \mathbb{R}, k \in \mathbb{N}$	$(-1)^k \frac{d^k}{ds^k}\left(\frac{s}{s^2+\omega^2}\right), \quad s > 0$
$t^k\sin(\omega t), \ \omega \in \mathbb{R}, k \in \mathbb{N}$	$(-1)^k \frac{d^k}{ds^k}\left(\frac{\omega}{s^2+\omega^2}\right), \quad s > 0$
$e^{\sigma t}\cos(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{s-\sigma}{(s-\sigma)^2+\omega^2}, \quad s > \sigma$
$e^{\sigma t}\sin(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{\omega}{(s-\sigma)^2+\omega^2}, \quad s > \sigma$
$t\, e^{\sigma t}\cos(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{(s-\sigma)^2-\omega^2}{((s-\sigma)^2+\omega^2)^2}, \quad s > \sigma$
$t\, e^{\sigma t}\sin(\omega t), \ \omega \in \mathbb{R}, \sigma \in \mathbb{R}$	$\frac{2(s-\sigma)\omega}{((s-\sigma)^2+\omega^2)^2}, \quad s > \sigma$
$u_a(t), \ a \geq 0$	$\frac{e^{-as}}{s}, \quad s > 0$