1. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ contained in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$, oriented outward, and let $C = \partial S$ be the boundary curve of S with the induced orientation (thus, C is a simple closed curve consisting of three arcs).

Let $\mathbf{F}(x, y, z) = (y, -x, z)$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ in three ways:

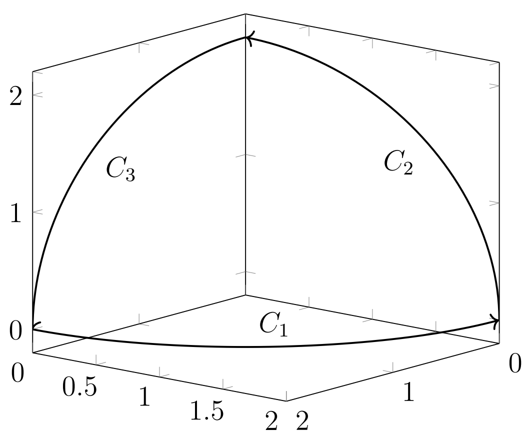
(a) Directly compute $\int_C \mathbf{F} \cdot \mathbf{dr}$. (This is equal to $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ by Stokes' theorem.)

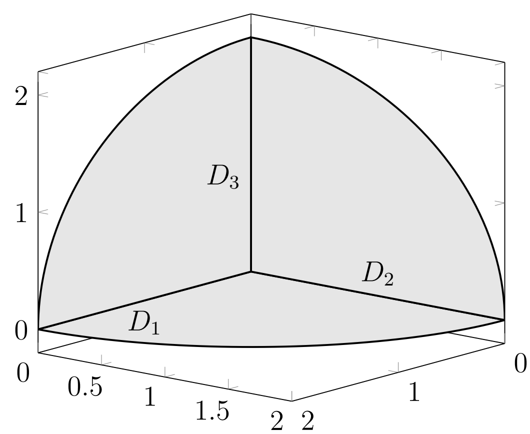
(b) Directly compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ by parametrizing S.

(c) Let R be the region enclosed by S and the three coordinate planes. Let D_1, D_2, D_3 be the three quarter-disks making up the boundary of R along with S, each oriented outward (we have $\partial R = S + D_1 + D_2 + D_3$). Explain why

$$\iint_{D_1+D_2+D_3} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} = -\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS},$$

and directly compute the flux of $\operatorname{curl} \mathbf{F}$ across each of the D_i, hence across S.





Solution.

(a) Write $C = \partial S = C_1 + C_2 + C_3$, where the C_i are the three arcs drawn above.

C_1: Parametrize this as

$$t \mapsto (2\cos(t), 2\sin(t), 0), \quad t \in \left[0, \frac{\pi}{2}\right].$$

The velocity is $\mathbf{v}(t) = (-2\sin(t), 2\cos(t), 0)$. We have

$$\mathbf{F}(\mathbf{r}(t)) = (2\sin(t), -2\cos(t), 0),$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = -4\sin^2(t) - 4\cos^2(t) = -4.$$

Therefore,

$$\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_0^{\pi/2} -4\, dt = -2\pi.$$

C_2: Parametrize this as

$$t \mapsto (0,\, 2\cos(t),\, 2\sin(t)), \quad t \in \left[0, \frac{\pi}{2}\right].$$

The velocity is $\mathbf{v}(t) = (0, -2\sin(t), 2\cos(t))$. We have

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= (2\cos(t),\, 0,\, 2\sin(t)), \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) &= 0 + 0 + 4\cos(t)\sin(t). \end{aligned}$$

Therefore,

$$\int_{C_2} \mathbf{F} \cdot \mathbf{dr} = \int_0^{\pi/2} 4\cos(t)\sin(t)\, dt = \left[2\sin^2(t)\right]_0^{\pi/2} = 2.$$

C_3: Parametrize this as

$$t \mapsto (-2\cos(t),\, 0,\, 2\sin(t)), \quad t \in \left[\frac{\pi}{2}, \pi\right].$$

The velocity is $\mathbf{v}(t) = (2\sin(t), 0, 2\cos(t))$. We have

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= (0,\, 2\sin(t),\, 2\sin(t)), \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) &= 0 + 0 + 4\cos(t)\sin(t). \end{aligned}$$

Therefore,

$$\int_{C_3} \mathbf{F} \cdot \mathbf{dr} = \int_{\pi/2}^{\pi} 4\cos(t)\sin(t)\, dt = \left[2\sin^2(t)\right]_{\pi/2}^{\pi} = -2.$$

Combining these computations, we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} = \int_{\partial S} \mathbf{F} \cdot \mathbf{dr} = -2\pi + 2 - 2 = -2\pi.$$

(b) Computing the curl, we have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \det \begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{pmatrix} \\ &= (0,\, 0,\, -1 - 1) \\ &= (0,\, 0,\, -2). \end{aligned}$$

Using the usual parametrization of the sphere,

$$(\phi, \theta) \mapsto (2\cos(\theta)\sin(\phi),\, 2\sin(\theta)\sin(\phi),\, 2\cos(\phi)), \quad \theta \in \left[0, \frac{\pi}{2}\right],\ \phi \in \left[0, \frac{\pi}{2}\right].$$

$$\begin{aligned} \mathbf{T}_\phi &= (2\cos(\theta)\cos(\phi),\, 2\sin(\theta)\cos(\phi),\, -2\sin(\phi)), \\ \mathbf{T}_\theta &= (-2\sin(\theta)\sin(\phi),\, 2\cos(\theta)\sin(\phi),\, 0), \\ \mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta &= (4\sin^2(\phi)\cos(\theta),\, 4\sin^2(\phi)\sin(\theta),\, 4\cos(\phi)\sin(\phi)). \end{aligned}$$

The normal points away from the origin, as required, so

$$\operatorname{curl}\mathbf{F}(\sigma(\phi,\theta))\cdot\mathbf{N}(\phi,\theta) = -8\cos(\phi)\sin(\phi).$$

Then,

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} &= \int_0^{\pi/2}\int_0^{\pi/2} -8\cos(\phi)\sin(\phi)\,d\phi\,d\theta \\ &= \int_0^{\pi/2}\left[-4\sin^2(\phi)\right]_0^{\pi/2}\,d\theta \\ &= \int_0^{\pi/2} -4\,d\theta \\ &= -2\pi,\end{aligned}$$

as in part (a).

(c) By the divergence theorem,

$$\iint_{\partial R}\operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = \iiint_R \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0.$$

So

$$\iint_{D_1+D_2+D_3}\operatorname{curl}\mathbf{F}\cdot\mathbf{dS} + \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = 0$$

and it follows that

$$\iint_{D_1+D_2+D_3}\operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = -\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS}.$$

Another argument: $D_1 + D_2 + D_3$ and S have the same boundary curve, but induce opposite orientations on it. It follows that

$$\iint_{D_1+D_2+D_3}\operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = -\int_{\partial S}\mathbf{F}\cdot\mathbf{dr} = -\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS}.$$

Now, because $\operatorname{curl}\mathbf{F} = -2\mathbf{e_z}$, we expect that there is no flux of $\operatorname{curl}\mathbf{F}$ through D_2 and D_3, but let's check this rigorously.

D_1: Parametrize as

$$(r,\theta)\mapsto(r\cos(\theta),\, r\sin(\theta),\, 0),\quad r\in[0,2],\ \theta\in\left[0,\frac{\pi}{2}\right].$$

We find that

$$\begin{aligned}\mathbf{T}_r &= (\cos(\theta),\, \sin(\theta),\, 0),\\ \mathbf{T}_\theta &= (-r\sin(\theta),\, r\cos(\theta),\, 0),\\ \mathbf{N} = \mathbf{T}_r\times\mathbf{T}_\theta &= (0,\, 0,\, r).\end{aligned}$$

This points up, so we take $-\mathbf{N} = (0, 0, -r)$ as the normal. We have $\operatorname{curl}\mathbf{F}(\sigma(r,\theta)) \cdot (0, 0, -r) = 2r$, so

$$\begin{aligned}\int_0^{\pi/2}\int_0^2 2r\,dr\,d\theta &= \int_0^{\pi/2}\left[r^2\right]_0^2\,d\theta\\ &= \int_0^{\pi/2} 4\,d\theta\\ &= 2\pi.\end{aligned}$$

D_2: Parametrize as

$$(r,\theta) \mapsto (0,\, r\cos(\theta),\, r\sin(\theta)) \quad r \in [0,2],\ \theta \in \left[0, \frac{\pi}{2}\right].$$

We have

$$\begin{aligned}\mathbf{T}_r &= (0,\, \cos(\theta),\, \sin(\theta)),\\ \mathbf{T}_\theta &= (0,\, -r\sin(\theta),\, r\cos(\theta)),\\ \mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta &= (r,\, 0,\, 0)\end{aligned}$$

The normal goes the wrong way (not that it matters for this part!), so we need to take $(-r, 0, 0)$. Now,

$$\operatorname{curl}\mathbf{F}(r,\,\theta)\cdot(-r,\,0,\,0) = 0.$$

So,

$$\iint_{D_2} \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = 0.$$

D_3: Parametrize as

$$(r,\theta) \mapsto (r\cos(\theta),\, 0,\, r\sin(\theta)) \quad r \in [0,2],\ \theta \in \left[0, \frac{\pi}{2}\right].$$

We have

$$\begin{aligned}\mathbf{T}_r &= (\cos(\theta),\, 0,\, \sin(\theta)),\\ \mathbf{T}_\theta &= (-r\sin(\theta),\, 0,\, r\cos(\theta)),\\ \mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta &= (0,\, -r,\, 0)\end{aligned}$$

The normal points the right way, but we have $\operatorname{curl}\mathbf{F}(r,\theta)\cdot(0,\,-r,\,0) = 0$, and so

$$\iint_{D_3} \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = 0.$$

We have

$$\iint_{D_1+D_2+D_3} \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = 2\pi + 0 + 0 = 2\pi,$$

so that

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = -\iint_{D_1+D_2+D_3} \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = -2\pi,$$

in agreement with parts (a) and (b).

2. Let R be the solid cube with vertices at $(\pm 2, \pm 2, \pm 2)$ (so, R has side length 4 and is centered at the origin). Let S be the boundary ∂R of R with the disk $x^2 + y^2 \le 1, z = 2$ removed from its top face. Let

$$\mathbf{F}(x, y, z) = (xy^2 + \arctan(y^2 z),\ y^3 - e^{-x(z-2)^2},\ \cos(\pi z)).$$

(a) Apply the divergence theorem to compute the flux of $\mathbf{F}$ through ∂R.

(b) Directly compute the flux of $\mathbf{F}$ through the disk $x^2 + y^2 \le 1, z = 2$, with the normal pointing up (that is, along the positive z-direction).

(c) Combine the results of parts (a) and (b) to compute the flux through S, with the normal pointing outward.

Solution.

(a) We have

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(xy^2 + \arctan(y^2 z)) + \frac{\partial}{\partial y}(y^3 - e^{-x(z-2)^2}) + \frac{\partial}{\partial z}(\cos(\pi z)) \\
&= y^2 + 0 + 3y^2 - 0 - \pi \sin(\pi z) \\
&= 4y^2 - \pi \sin(\pi z).
\end{aligned}$$

The integral of divergence over the cube R is

$$\begin{aligned}
\iiint_R \operatorname{div} \mathbf{F}\, dV &= \int_{-2}^{2} \int_{-2}^{2} \int_{-2}^{2} 4y^2 - \pi \sin(\pi z)\, dz\, dy\, dx \\
&= \int_{-2}^{2} \int_{-2}^{2} 16y^2 + [\cos(\pi z)]_{-2}^{2}\, dy\, dx \\
&= \int_{-2}^{2} \int_{-2}^{2} 16y^2 + 0\, dy\, dx \\
&= \int_{-2}^{2} \left[\frac{16}{3} y^3\right]_{-2}^{2} dx \\
&= \frac{1024}{3}.
\end{aligned}$$

(b) We parametrize as

$$(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta), 2) \quad r \in [0,1],\ \theta \in [0, 2\pi].$$

We have

$$\begin{aligned}
\mathbf{T}_r &= (\cos(\theta), \sin(\theta), 0), \\
\mathbf{T}_\theta &= (-r\sin(\theta), r\cos(\theta), 0), \\
\mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta &= (0, 0, r).
\end{aligned}$$

The normal points upward, as needed. Then,

$$\begin{aligned} \mathbf{F}(\sigma(r,\theta)) &= (*,\ *,\ \cos(2\pi)) = (*,\ *,\ 1) \\ \mathbf{F}(\sigma(r,\theta)) \cdot \mathbf{N}(r,\theta) &= r. \end{aligned}$$

So, the flux of $\mathbf{F}$ through the disk is

$$\iint_{\text{Disk}} \mathbf{F} \cdot \mathbf{dS} = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \pi.$$

(c) The flux of $\mathbf{F}$ through S is

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_{\partial R} \mathbf{F} \cdot \mathbf{dS} - \iint_{\text{Disk}} \mathbf{F} \cdot \mathbf{dS} = \frac{1024}{3} - \pi.$$

3. (a) The intersection of the (hollow) cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 4$ consists of two pieces, each of which is a simple closed curve. Parametrize the piece with $z \geq 0$. (Suggestion: First parametrize its shadow in the xy-plane.)

(b) If $\mathbf{G}(x,y,z) = (x^2 z,\ y^2 x,\ -z^2 x)$ and $\mathbf{F}(x,y,z) = (0,\ x^2 + z^2,\ y^2)$, check that $\operatorname{curl} \mathbf{G} = \mathbf{F}$.

(c) Let S be the surface described by $x^2 + y^2 \leq 1,\ y^2 + z^2 = 4,\ z \geq 0$. Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{dS}$, with normals pointing out of the cylinder $y^2 + z^2 = 4$ (that is, with $\mathbf{N} \cdot \mathbf{e_z} \geq 0$).

Possibly useful identities: $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}, \quad \sin^2(x) = \dfrac{1 - \cos(2x)}{2}$.

Solution.

(a) The shadow of the intersection in the xy-plane is a circle of radius 1, which we can parametrize by $t \mapsto (\cos(t), \sin(t), 0),\ t \in [0, 2\pi]$. Then, since every point lies on the cylinder $y^2 + z^2 = 4$, we know that $z = \pm\sqrt{4 - y^2}$. Finally, since $z \geq 0$, $z = \sqrt{4 - y^2}$. Thus, the intersection curve can be parametrized as

$$t \mapsto \left(\cos(t),\ \sin(t),\ \sqrt{4 - \sin^2(t)}\right), \quad t \in [0, 2\pi].$$

(b) We have

$$\begin{aligned} \operatorname{curl} \mathbf{G} &= \det \begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & y^2 x & -z^2 x \end{pmatrix} \\ &= (0 - 0,\ -(-z^2 - x^2),\ y^2 - 0) \\ &= (0,\ x^2 + z^2,\ y^2) \\ &= \mathbf{F}. \end{aligned}$$

(c) By Stokes' theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \operatorname{curl} \mathbf{G} \cdot \mathbf{dS} = \int_{\partial S} \mathbf{G} \cdot \mathbf{dr},$$

so it is equivalent to compute $\int_{\partial S} \mathbf{G} \cdot \mathbf{dr}$. The boundary curve of S is precisely the curve parametrized in part (a) (and the parametrization found in (a) has the correct orientation).

The velocity of the parametrization is

$$\mathbf{v}(t) = \left(-\sin(t), \cos(t), \frac{-\cos(t)\sin(t)}{\sqrt{4 - \sin^2(t)}} \right).$$

We have

$$\begin{aligned} \mathbf{G}(\mathbf{r}(t)) &= \left(\cos^2(t)\sqrt{4 - \sin^2(t)}, \sin^2(t)\cos(t), -\cos(t)(4 - \sin^2(t)) \right), \\ \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{v}(t) &= -\sin(t)\cos^2(t)\sqrt{4 - \sin^2(t)} + \sin^2(t)\cos^2(t) + \cos^2(t)\sin(t)\frac{4 - \sin^2(t)}{\sqrt{4 - \sin^2(t)}} \\ &= \sin^2(t)\cos^2(t), \end{aligned}$$

since $\dfrac{4 - \sin^2(t)}{\sqrt{4 - \sin^2(t)}} = \sqrt{4 - \sin^2(t)}$.

Now, we use the suggested identities to find that

$$\begin{aligned} \sin^2(t)\cos^2(t) &= \frac{1 - \cos(2t)}{2} \frac{1 + \cos(2t)}{2} \\ &= \frac{1 - \cos^2(2t)}{4} \\ &= \frac{1 - (1 + \cos(4t))/2}{4} \\ &= \frac{1 - \cos(4t)}{8}. \end{aligned}$$

So that

$$\begin{aligned} \int \mathbf{G} \cdot \mathbf{dr} &= \int_0^{2\pi} \sin^2(t)\cos^2(t)\, dt \\ &= \int_0^{2\pi} \frac{1 - \cos(4t)}{8}\, dt \\ &= \frac{\pi}{4} - \left[\frac{\sin(4t)}{32} \right]_0^{2\pi} \\ &= \frac{\pi}{4}. \end{aligned}$$

4. If S is a closed surface bounding a region R, and $\mathbf{F}$ is a vector field defined everywhere in R, then by the divergence theorem we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = \iiint_R \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0,$$

since $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ identically. This is analogous to the fact that $\int_C \nabla f\cdot\mathbf{dr} = 0$ for a closed curve.

Let $\mathbf{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$ with $(x,y,z)\neq(0,0,0)$. Check that $\operatorname{div}\mathbf{F} = 0$ everywhere $\mathbf{F}$ is defined, but that $\iint_S \mathbf{F}\cdot\mathbf{dS} = 4\pi$, where S is the unit sphere $x^2+y^2+z^2=1$.

Solution. We have

$$\begin{aligned}\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} &= \frac{1\cdot(x^2+y^2+z^2)^{3/2} - x\,\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot(2x)}{(x^2+y^2+z^2)^3}\\ &= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}\\ &= \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Similar computations also give

$$\begin{aligned}\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} &= \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}},\\ \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} &= \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Therefore,

$$\operatorname{div}\mathbf{F} = \frac{(y^2+z^2-2x^2)+(x^2+z^2-2y^2)+(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{5/2}} = 0.$$

We check that in fact for any sphere S of radius a centered at the origin, $\iint_S \mathbf{F}\cdot\mathbf{dS} = 4\pi$.

Parametrize by

$$(\phi,\,\theta)\mapsto(a\cos(\theta)\sin(\phi),\,a\sin(\theta)\sin(\phi),\,a\cos(\phi)),\quad \phi\in[0,\pi],\ \theta\in[0,2\pi].$$

We have

$$\begin{aligned}\mathbf{T}_\phi &= (a\cos(\theta)\cos(\phi),\,a\sin(\theta)\cos(\phi),\,-a\sin(\phi)),\\ \mathbf{T}_\theta &= (-a\sin(\theta)\sin(\phi),\,a\cos(\theta)\sin(\phi),\,0),\\ \mathbf{N} = \mathbf{T}_\phi\times\mathbf{T}_\theta &= (a^2\cos(\theta)\sin^2(\phi),\,a^2\sin(\theta)\sin^2(\phi),\,a^2\cos(\phi)\sin(\phi)).\end{aligned}$$

Note that for points on the sphere, $(x^2+y^2+z^2)^{3/2} = a^3$ holds. So we have

$$\begin{aligned}
\mathbf{F}(\sigma(\phi,\theta)) &= \left(\frac{a\cos(\theta)\sin(\phi)}{a^3}, \frac{a\sin(\theta)\sin(\phi)}{a^3}, \frac{a\cos(\phi)}{a^3}\right) \\
&= \left(\frac{\cos(\theta)\sin(\phi)}{a^2}, \frac{\sin(\theta)\sin(\phi)}{a^2}, \frac{\cos(\phi)}{a^2}\right), \\
\mathbf{F}(\sigma(\phi,\theta))\cdot\mathbf{N}(\phi,\theta) &= \cos^2(\theta)\sin^3(\phi) + \sin^2(\theta)\sin^3(\phi) + \cos^2(\phi)\sin(\phi) \\
&= \sin^3(\phi) + \cos^2(\phi)\sin(\phi) \\
&= \sin(\phi)\left(\sin^2(\phi)+\cos^2(\phi)\right) \\
&= \sin(\phi).
\end{aligned}$$

Finally, integrating over the sphere, we get

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{dS} &= \int_0^{2\pi}\int_0^{\pi} \sin(\phi)\,d\phi\,d\theta \\
&= \int_0^{2\pi} \left[-\cos(\phi)\right]_0^{\pi}\,d\theta \\
&= \int_0^{2\pi} 2\,d\theta \\
&= 4\pi.
\end{aligned}$$

Optional Problem. Define a 'toric change of variables' $T\colon D \to \mathbb{R}^3_{(x,y,z)}$, where

$$D = \{(r,\theta,t) \in \mathbb{R}^3 : 0 \le r < b,\ 0 \le t \le 2\pi,\ 0 \le \theta \le 2\pi\},$$

by

$$\begin{aligned}
x(r,\theta,t) &= (b + r\cos(t))\cos(\theta), \\
y(r,\theta,t) &= (b + r\cos(t))\sin(\theta), \\
z(r,\theta,t) &= r\sin(t).
\end{aligned}$$

(a) Find a region V^* in $\mathbb{R}^3_{(r,\theta,t)}$ such that its image $V = T(V^*)$ is the region bounded by the torus with radii a and b.

(b) Check that $\det \dfrac{\partial(x,y,z)}{\partial(r,\theta,t)} = r(b + r\cos(t))$.

(c) Apply the change of variables theorem to conclude that the integral

$$\iiint_{V^*} r(b + r\cos(t))\,dt\,dr\,d\theta$$

is equal to the volume of V, and compute the integral.

Solution.

(a) The inequalities $0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le t \le 2\pi$ describe V^*.

(b) We have

$$\begin{aligned}
\det \frac{\partial(x,y,z)}{\partial(r,\theta,t)} &= \det \begin{pmatrix} \cos(t)\cos(\theta) & -(b + r\cos(t))\sin(\theta) & -r\sin(t)\cos(\theta) \\ \cos(t)\sin(\theta) & (b + r\cos(t))\cos(\theta) & -r\sin(t)\sin(\theta) \\ \sin(t) & 0 & r\cos(t) \end{pmatrix} \\
&= \sin(t)\det \begin{pmatrix} -(b + r\cos(t))\sin(\theta) & -r\sin(t)\cos(\theta) \\ (b + r\cos(t))\cos(\theta) & -r\sin(t)\sin(\theta) \end{pmatrix} + \cdots \\
&\cdots + r\cos(t) \begin{pmatrix} \cos(t)\cos(\theta) & -(b + r\cos(t))\sin(\theta) \\ \cos(t)\sin(\theta) & (b + r\cos(t))\cos(\theta) \end{pmatrix} \\
&= \sin(t)(r\sin(t)(b + r\cos(t))) + r\cos(t)(\cos(t)(b + r\cos(t))) \\
&= r(b + r\cos(t)).
\end{aligned}$$

(c) The claimed integral is indeed equal to the volume of V by the change of variables theorem. We compute that

$$\begin{aligned}
\iiint_{V^*} r(b + r\cos(t))\, dt\, dr\, d\theta &= \int_0^{2\pi} \int_0^a \int_0^{2\pi} rb + r^2\cos(t)\, dt\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^a 2\pi rb + 0\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\pi r^2 b\right]_{r=0}^{r=a} d\theta \\
&= \int_0^{2\pi} \pi a^2 b\, d\theta \\
&= 2\pi^2 a^2 b.
\end{aligned}$$