MTHE 227 Problem Set 11 Solutions

1. As a reminder, a torus with radii a and b is the surface of revolution of the circle $(x - b)^2 + z^2 = a^2$ in the xz-plane about the z-axis (a and b are positive real numbers, with $b > a$). (For two pictures of a torus, see the last page of this problem set.)

(a) Find a function $f(r, \theta, z)$ and a constant $c \in \mathbb{R}$ so that the equation $f(r, \theta, z) = c$ in cylindrical coordinates describes the torus with radii a and b.

(b) Set up two triple integrals in cylindrical coordinates for the volume of the solid torus (the three-dimensional region bounded by a torus) with radii a and b : one with order of integration $dr\,dz\,d\theta$ and the other with order of integration $dz\,dr\,d\theta$.

(c) Check that the volume of the solid torus is equal to $(\pi a^2)(2\pi b) = 2\pi^2 a^2 b$. (It is only necessary to integrate using one of the orders of part (b).)

(You may need to make a sin / cos-type trigonometric substitution.)

Solution.

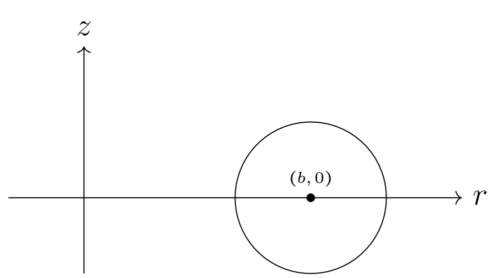
(a) As discussed briefly in class, to find the equation of a surface of revolution of the curve described by $f(x, z) = c$ in the xz-plane about the z-axis, we can simply replace x by r. Roughly speaking, the reason is that the cross-sections of a surface of revolution by half-planes described by $\theta = \theta_0$ all look like the original curve, with r playing the role of the x-coordinate. Applying this principle,

$$(r - b)^2 + z^2 = a^2$$

describes the torus with radii a and b (so, take $f(r, \theta, z) = (r - b)^2 + z^2$ and $c = a^2$).

Notice that $(x - b)^2 + z^2 = a^2$ describes a *cylinder* in $\mathbb{R}^3$, so converting to cylindrical coordinates using $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, we would obtain the equation of a cylinder in cylindrical coordinates and not a torus.

(b) A cross section of the torus by any half-plane described by $\theta = \theta_0$ is a circle of radius a centered at $(b, 0)$.



Order $dr\,dz\,d\theta$: Fixing θ, we have a cross-section as described above. Fixing z, we can solve $(r - b)^2 + z^2 = a^2$ to see that r goes between $b - \sqrt{a^2 - z^2}$ and $b + \sqrt{a^2 - z^2}$. Then, z ranges between $-a$ and a, and θ ranges between 0 and 2π. So the integral is

$$\int_0^{2\pi} \int_{-a}^{a} \int_{b-\sqrt{a^2-z^2}}^{b+\sqrt{a^2-z^2}} r\,dr\,dz\,d\theta.$$

Order $dz\,dr\,d\theta$: Fixing θ, we have a cross-section as above. Fixing r, solve $(r-b)^2+z^2 = a^2$ to see that z goes from $-\sqrt{a^2-(r-b)^2}$ to $\sqrt{a^2-(r-b)^2}$. Then, r ranges from $b-a$ to $b+a$, and θ ranges between 0 and 2π. The integral is

$$\int_0^{2\pi}\int_{b-a}^{b+a}\int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} r\,dz\,dr\,d\theta.$$

(c) Order $dr\,dz\,d\theta$: We have

$$\int_0^{2\pi}\int_{-a}^{a}\int_{b-\sqrt{a^2-z^2}}^{b+\sqrt{a^2-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_{-a}^{a}\frac{(b+\sqrt{a^2-z^2})^2-(b-\sqrt{a^2-z^2})^2}{2}\,dz\,d\theta$$
$$= \int_0^{2\pi}\int_{-a}^{a} 2b\sqrt{a^2-z^2}\,dz\,d\theta.$$

Make the substitution $z = a\sin(t)$, $dz = a\cos(t)dt$. Then $a^2-z^2 = a^2(1-\sin^2(t)) = a^2\cos^2(t)$. As t ranges from $-\pi/2$ to $\pi/2$, z ranges from $-a$ to a. So,

$$\int_{-a}^{a} 2b\sqrt{a^2-z^2}\,dz = \int_{-\pi/2}^{\pi/2} 2b\,a\cos(t)\,a\cos(t)\,dt = 2a^2b\int_{-\pi/2}^{\pi/2}\cos^2(t)\,dt = a^2b\pi.$$

Finally,

$$\int_0^{2\pi}\int_{-a}^{a} 2b\sqrt{a^2-z^2}\,dz\,d\theta = \int_0^{2\pi} a^2b\pi\,d\theta = 2\pi^2a^2b,$$

as claimed.

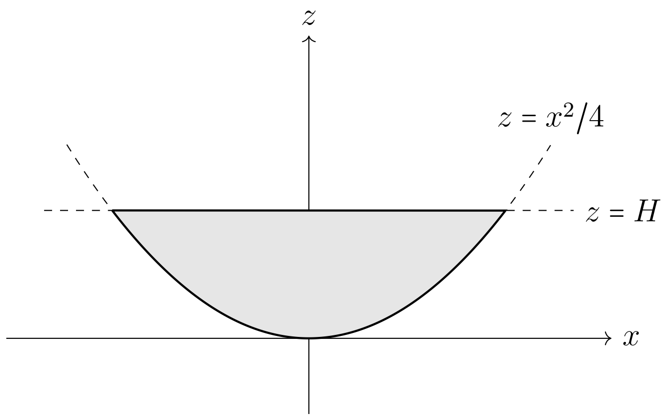
Order $dz\,dr\,d\theta$: We have

$$\int_0^{2\pi}\int_{b-a}^{b+a}\int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_{b-a}^{b+a} r\left(\sqrt{a^2-(r-b)^2}-(-\sqrt{a^2-(r-b)^2})\right)dr\,d\theta$$
$$= \int_0^{2\pi}\int_{b-a}^{b+a} 2r\sqrt{a^2-(r-b)^2}\,dr\,d\theta.$$

Now, make the substitution $r-b = a\cos(t)$, and proceed as in the previous part.

2. Find the volume of the region bounded by the surface $z = x^2/4$, and the three planes $y = 0$, $y = \ell$ and $z = H$ in $\mathbb{R}^3$, as a function of ℓ and H.

Solution. The cross-sections of the surface by plane parallel to the xz-plane are identical, which suggests taking the order of integration $dz\,dx\,dy$. The cross-sections look like



The parabola $z = x^2/4$ intersects the line $z = H$ when $x^2/4 = H$, or $x = \pm 2\sqrt{H}$. The required volume is equal to

$$\begin{aligned}
\int_0^\ell \int_{-2\sqrt{H}}^{2\sqrt{H}} \int_{x^2/4}^{H} dz\,dx\,dy &= \int_0^\ell \int_{-2\sqrt{H}}^{2\sqrt{H}} H - \frac{x^2}{4}\,dx\,dy \\
&= \int_0^\ell \left[Hx - \frac{x^3}{12}\right]_{-2\sqrt{H}}^{2\sqrt{H}} dy \\
&= \int_0^\ell 4H^{3/2} - \frac{8H^{3/2} - (-8H^{3/2})}{12}\,dy \\
&= H^{3/2}\ell\left(4 - \frac{4}{3}\right) \\
&= \frac{8}{3}H^{3/2}\ell
\end{aligned}$$

3. Imagine a pool of still fluid (in other words, the fluid is static and in equilibrium). Let h denote the vertical coordinate, measured down from the surface of the fluid, and let x and y denote the usual Cartesian coordinates. As you likely know, if the fluid is incompressible (this is true of water, to a good approximation), the pressure exerted by the fluid varies as[1]

$$p(h, y, z) = \delta g h,$$

where δ is the density of the fluid (assumed uniform), and g is the gravitational constant.

Because of the pressure difference at different heights, a region submerged in the fluid will have a net upward force on it, called the buoyant force, which may be computed as follows.

Let S be a closed (smooth, orientable) surface submerged in the fluid, bounding a region R, and choose inward pointing normals. A small piece of S around the point (x, y, h) with area ΔA will have a force directed perpendicular to it and equal in magnitude (to a good approximation) to $p(x, y, h)\,\Delta A$ (this is just the definition of pressure). To find its component directed up, we can compute the dot product

$$-\mathbf{e_h} \cdot \big((p(x,y,h)\Delta A)\ \hat{\mathbf{N}}(x,y,h)\big) = (-\delta g h\,\mathbf{e_h}) \cdot \hat{\mathbf{N}}(x,y,h)\Delta A$$

[1] Instructor's note: On the other hand, if you do not know why, and are curious why, ask me!

(the negative sign before $\mathbf{e}_h$ is necessary because of the convention that h points down).

Defining the vector field

$$\mathbf{B}(x, y, h) := (0, 0, -\delta g h) = -\delta g h\, \mathbf{e}_h,$$

and taking $\Delta A \to 0$, the buoyant force on S is therefore equal to the integral

$$\text{Buoyant Force} = \iint_S \mathbf{B} \cdot \hat{\mathbf{N}}\, dS = \iint_S \mathbf{B} \cdot \mathbf{dS}.$$

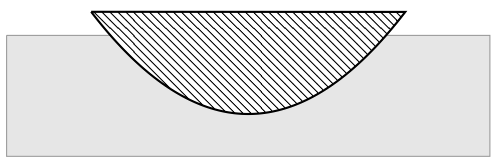
(d) Prove the following theorem, applying the divergence theorem:

Theorem (Archimedes)**.** *The buoyant force on S is equal to the weight of the fluid displaced by S.*

(Take care with the orientation of $\hat{\mathbf{N}}$. In the statement, weight is the product of mass and the gravitational constant g.)

(e) Justify using (b): If R is a region of uniform density d placed in the pool, it will rise if $d < \delta$ and sink if $d > \delta$.

Optional Problem. Let R be a ship modeled as a solid of the kind looked at in Problem 2, of mass $1,080,000\,\text{kg}$, length $\ell = 30\,\text{m}$ and height $H = 10\,\text{m}$. Take the fluid to be water (so, with density $\delta = 1,000\,\text{kg/m}^3$). When the ship is floating at the surface of the water, where will the water level be (measured from the bottom of the ship)?



Solution.

(a) One clarifying remark that should be made is that the flux integral $\iint_S \mathbf{B} \cdot \mathbf{dS}$, being a number, only captures the coefficient of the buoyant force along $-\mathbf{e_h}$ (or, in other words, along $\mathbf{e_z}$ — the force points up when the integral is positive).

Treating (x, y, h)-coordinates as usual Cartesian coordinates, the divergence of $\mathbf{B}$ is

$$\operatorname{div} \mathbf{B} = -\delta g.$$

This is a little subtle. It should be pointed out that in more general coordinate systems the expression for the divergence in terms of the components of a vector field may not be as simple. For instance, the general expression for the divergence in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.$$

This is different from $\dfrac{\partial F_r}{\partial r} + \dfrac{\partial F_\theta}{\partial \theta} + \dfrac{\partial F_z}{\partial z}$, the expression which may be one's first guess. The key is that the direction vectors in (x, y, h)-coordinates are constant, just as they are for Cartesian coordinates.

Because we chose inward-pointing normals in the derivation of the flux integral for the buoyant force, and the divergence theorem applies for outward-pointing normals, we need to introduce a negative sign in applying the divergence theorem.

Therefore,

$$\iint_S \mathbf{B} \cdot \mathbf{dS} = -\iiint_R \operatorname{div} \mathbf{B} \, dV = -\Big(\iiint_R -\delta g \, dV\Big) = g \iiint_R \delta \, dV,$$

and the integral $\iiint_R \delta \, dV$ is exactly equal to the mass of water displaced by R (or, as loosely put in the problem statement, by S), hence $g \iiint_R \delta \, dV$ is the total weight displaced by R.

(b) (Of course, we are assuming the only forces acting on R are gravity and the buoyant force due to the fluid!)

The force of gravity on the region R is equal to its weight directed along $\mathbf{e_h}$, or $g \iiint_R d \, dV \, \mathbf{e_h}$; the buoyant force on R is equal to the weight of the fluid displaced by R directed along $-\mathbf{e_h}$, or $-g \iiint_R \delta \, dV \, \mathbf{e_h}$. The net force on R is therefore equal to

$$\left(g(\delta - d) \iiint_R dV\right)(-\mathbf{e_h}) = \left(g(\delta - d) \iiint_R dV\right)\mathbf{e_z}.$$

The net force is directed along $\mathbf{e_z}$ when $\delta > d$ and along $-\mathbf{e_z}$ when $\delta < d$, hence the region will rise in the first case and sink in the second.

Solution of the Optional Problem. The magnitude of the buoyant force will be equal (by part (a)) to the weight of water displaced by the submerged part of the ship. If χ denotes the water level (meaning the height from the bottom of the ship to the surface of the water), then the volume displaced by the submerged part is equal to (by Problem 2)

$$\frac{8}{3}\chi^{3/2}30.$$

The buoyant force on the ship due to the submerged part is then

$$g \cdot 80 \cdot \chi^{3/2} \cdot 1000 = 80,000 g \, \chi^{3/2}.$$

On the other hand, the force of gravity on the ship is equal to

$$g \cdot 1,080,000$$

(pointing down). When the ship floats, the net force should be zero, so

$$80,000 g \, \chi^{3/2} = 1,080,000 g.$$

Solving for χ, we obtain

$$\chi = \left(\frac{1,080,000}{80,000}\right)^{2/3} = \frac{27^{2/3}}{2^{2/3}} = \frac{9}{2^{2/3}} \approx 5.67 \text{ m}.$$

4. Let R be the region $1 \le x^2 + y^2 \le 9,\ 0 \le z \le 2$ in $\mathbb{R}^3$, and let S be its boundary surface, oriented outward from R. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (2x,\ xy^2,\ xyz).$$

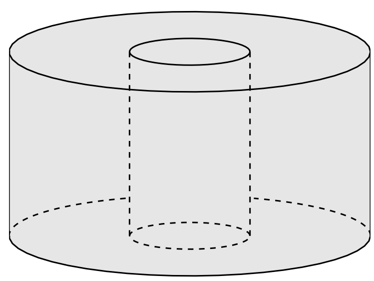
(a) Sketch R. Notice that the boundary surface S splits into four pieces.

(b) Compute the flux integral $\iint_S \mathbf{F} \cdot \mathbf{dS}$ directly, by parametrizing each of the four pieces and computing the flux of $\mathbf{F}$ across each.

(c) Compute $\operatorname{div} \mathbf{F}$, and compute the triple integral $\iiint_R \operatorname{div} \mathbf{F}\, dV$ directly. The answer should be equal to that of part (b) by the divergence theorem.

Solution.

(a) The region is a cylinder of radius 3 and height 2, with a smaller coaxial cylinder of radius 1 removed. There are four parts to the boundary: two annular disks on the top and bottom (taking the z-axis as the vertical), and an inner and outer wall.



(b) We are asked to set up and compute flux integrals through four pieces of the boundary ∂R.

First piece: The top disk of R, oriented upwards (that is, along the $\mathbf{e_z}$ direction).

We can parametrize as

$$(u, v) \mapsto (u\cos(v),\ u\sin(v),\ 2), \qquad u \in [1, 3],\ v \in [0, 2\pi].$$

We find the normal

$$\begin{aligned}
\mathbf{T}_u &= (\cos(v), \sin(v), 0), \\
\mathbf{T}_v &= (-u\sin(v),\ u\cos(v), 0), \\
\mathbf{T}_u \times \mathbf{T}_v &= \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{pmatrix} \\
&= (0,\ 0,\ u).
\end{aligned}$$

This points upward, as needed.

Now, to compute the flux,

$$\begin{aligned}\mathbf{F}(\sigma(u,v)) &= (2u\cos(v),\, u^3\cos(v)\sin^2(v),\, 2u^2\cos(v)\sin(v)),\\ \mathbf{F}(\sigma(u,v))\cdot\mathbf{N}(u,v) &= 0+0+2u^3\cos(v)\sin(v).\end{aligned}$$

So that

$$\begin{aligned}\iint_{\text{top disk}} \mathbf{F}\cdot\mathbf{dS} &= \int_1^3\int_0^{2\pi} 2u^3\cos(v)\sin(v)\,dv\,du\\ &= \int_1^3 \left[u^3\sin^2(v)\right]_0^{2\pi}\,du\\ &= 0.\end{aligned}$$

Second piece: The bottom disk of R, oriented downwards (along $-\mathbf{e_z}$).

We can parametrize as

$$(u,v)\mapsto(u\cos(v),\, u\sin(v),\, 0),\qquad u\in[1,3],\ v\in[0,2\pi].$$

Because the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are the same as the top disk, we obtain the normal $\mathbf{N}(u,v)=(0,\,0,\,u)$. Since we need to choose the downward normal, however, we take $\mathbf{N}(u,v)=(0,\,0,\,-u)$.

To compute the flux,

$$\begin{aligned}\mathbf{F}(\sigma(u,v)) &= (2u\cos(v),\, u^3\cos(v)\sin^2(v),\, 0),\\ \mathbf{F}(\sigma(u,v))\cdot\mathbf{N}(u,v) &= 0+0+0=0.\end{aligned}$$

Therefore,

$$\iint_{\text{bottom disk}} \mathbf{F}\cdot\mathbf{dS} = 0.$$

Third piece: The outer cylinder wall, oriented outward.

Can parametrize as

$$(u,v)\mapsto(3\cos(u),\, 3\sin(u),\, v),\quad u\in[0,2\pi],\ v\in[0,2].$$

We find the normal

$$\begin{aligned}\mathbf{T}_u &= (-3\sin(u),\, 3\cos(u),\, 0)\\ \mathbf{T}_v &= (0,\,0,\,1)\\ \mathbf{T}_u\times\mathbf{T}_v &= \det\begin{pmatrix}\mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z}\\ -3\sin(u) & 3\cos(u) & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= (3\cos(u),\, 3\sin(u),\, 0).\end{aligned}$$

This points outward, as needed.

Computing the flux,

$$\begin{aligned}
\mathbf{F}(\sigma(u,v)) &= (6\cos(u),\, 27\cos(u)\sin^2(u),\, 3v\cos(u)\sin(u)) \\
\mathbf{F}(\sigma(u,v)) \cdot \mathbf{N}(u,v) &= 18\cos^2(u) + 81\cos(u)\sin^3(u) + 0.
\end{aligned}$$

So that

$$\begin{aligned}
\iint_{\text{outer wall}} \mathbf{F} \cdot \mathbf{dS} &= \int_0^2 \int_0^{2\pi} 18\cos^2(u) + 81\cos(u)\sin^3(u)\, du\, dv \\
&= \int_0^2 18\pi + 81\left[\frac{1}{4}\sin^4(u)\right]_0^{2\pi} dv \\
&= 36\pi.
\end{aligned}$$

Four piece: The inner cylinder wall, oriented inward.

Can parametrize as

$$(u,v) \mapsto (\cos(u),\, \sin(u),\, v), \quad u \in [0, 2\pi],\ v \in [0,2].$$

For the normal,

$$\begin{aligned}
\mathbf{T}_u &= (-\sin(u),\, \cos(u),\, 0), \\
\mathbf{T}_v &= (0,\, 0,\, 1), \\
\mathbf{T}_u \times \mathbf{T}_v &= \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= (\cos(u),\, \sin(u),\, 0).
\end{aligned}$$

This points outward, so for the inner-pointing normal we take $\mathbf{N} = (-\cos(u), -\sin(u), 0)$. We have

$$\begin{aligned}
\mathbf{F}(\sigma(u,v)) &= (2\cos(u),\, \cos(u)\sin^2(u),\, v\cos(u)\sin(u)), \\
\mathbf{F}(\sigma(u,v)) \cdot \mathbf{N}(u,v) &= -2\cos^2(u) - \cos(u)\sin^3(u) + 0.
\end{aligned}$$

The flux integral is

$$\begin{aligned}
\iint_{\text{inner wall}} \mathbf{F} \cdot \mathbf{dS} &= \int_0^2 \int_0^{2\pi} -2\cos^2(u) - \cos(u)\sin^3(u)\, du\, dv \\
&= \int_0^2 -2\pi - \left[\frac{1}{4}\sin^4(u)\right]_0^{2\pi} dv \\
&= -4\pi.
\end{aligned}$$

So that, putting it all together,

$$\iint_{\partial R} \mathbf{F} \cdot \mathbf{dS} = 0 + 0 + 36\pi - 4\pi = 32\pi.$$

(c) We have

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(xyz) = 2 + 2xy + xy = 2 + 3xy.$$

To integrate over R, it is convenient to switch to cylindrical coordinates. The integrand is

$$2 + 3(r\cos(\theta))(r\sin(\theta)) = 2 + 3r^2\cos(\theta)\sin(\theta)$$

and R is described by the inequalities

$$1 \le r \le 3, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2.$$

Because of the $\cos(\theta)\sin(\theta)$ term, it turns out to be good to integrate with respect to θ first.

$$\begin{aligned}
\int_1^3\int_0^2\int_0^{2\pi}\left(2 + 3r^2\cos(\theta)\sin(\theta)\right) r\,d\theta\,dz\,dr &= \int_1^3\int_0^2\left[2r + \frac{3}{2}r^3\sin^2(\theta)\right]_0^{2\pi} dz\,dr \\
&= \int_1^3\int_0^2 4\pi r + 0\,dzdr \\
&= \int_1^3 [4\pi rz]_0^2\,dr \\
&= \int_1^3 8\pi r\,dr \\
&= \left[4\pi r^2\right]_1^3 \\
&= 32\pi,
\end{aligned}$$

so we find that

$$\iint_{\partial R}\mathbf{F}\cdot\mathbf{dS} = \iiint_R \operatorname{div}(\mathbf{F})\,dV = 32\pi,$$

verifying the divergence theorem.

We could have also argued that the integral of xy over R must vanish by symmetry.

5. As a reminder, spherical coordinates on $\mathbb{R}^3$ are given by the following map $D \to \mathbb{R}^3_{(x,y,z)}$:

$$\begin{aligned}
x(\rho,\theta,\phi) &= \rho\cos(\theta)\sin(\phi), \\
y(\rho,\theta,\phi) &= \rho\sin(\theta)\sin(\phi), \\
z(\rho,\theta,\phi) &= \rho\cos(\phi),
\end{aligned}$$

where

$$D = \{(\rho,\theta,\phi) \in \mathbb{R}^3 : \rho \ge 0,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}.$$

Check that

$$\left|\det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| := \left|\det\begin{pmatrix}\partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi\end{pmatrix}\right| = \rho^2\sin(\phi),$$

where $|\cdot|$ denotes the absolute value.

Solution. We have

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det \begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{pmatrix}$$

Expanding along the bottom row, this is equal to

$$\begin{aligned}
&\cos(\phi)\det\begin{pmatrix} -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \end{pmatrix} - 0 + (-\rho\sin(\phi))\det\begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) \end{pmatrix} \\
&= \cos(\phi)\left(-\rho^2\sin^2(\theta)\sin(\phi)\cos(\phi) - \rho^2\cos^2(\theta)\sin(\phi)\cos(\phi)\right) - \cdots \\
&\cdots - \rho\sin(\phi)\left(\rho\cos^2(\theta)\sin^2(\phi) + \rho\sin^2(\theta)\sin^2(\phi)\right) \\
&= -\rho^2\sin(\phi)\cos^2(\phi) - \rho^2\sin^3(\phi) \\
&= -\rho^2\sin(\phi)\left(\cos^2(\phi) + \sin^2(\phi)\right) \\
&= -\rho^2\sin(\phi).
\end{aligned}$$

Taking the absolute value, we see that

$$\left|\det\begin{pmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi \end{pmatrix}\right| = \rho^2\sin(\phi).$$

As mentioned in class, the minus sign shows up because of the choice of order of the spherical coordinates. We would have

$$\det\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin(\phi).$$