1. (a) Sketch the cross-section of the (hollow) cylinder $y^2 + z^2 = 4$ in the xz-plane, as well as the vector field

$$\mathbf{F}(x,y,z) = \begin{cases} \left(1 - \frac{y^2+z^2}{4}, 0, 0\right), & y^2 + z^2 < 4 \\ \mathbf{0}, & \text{Otherwise} \end{cases}.$$

in this cross-section.

This is a simple model of water flowing through a pipe without turbulence (interestingly, the velocity goes to zero at the boundary!).

(b) The disk S described by $x = x_0,\ y^2 + z^2 \le 4$ (the region bounded by the cross-section of the pipe in the plane $x = x_0$) may be parametrized by

$$(u, v) \mapsto (x_0, u\cos(v), u\sin(v)), \qquad u \in [0,2],\ v \in [0, 2\pi].$$

Find the flux $\iint_S \mathbf{F} \cdot \mathbf{dS}$ of the vector field $\mathbf{F}$ of part (a) through S, with the normal pointing along $\mathbf{e}_x$ (that is, along the positive x direction).

(c) The portion Σ of the cylinder $y^2 + z^2 = 1$ between $x = 1$ and $x = 2$ may be parametrized by

$$(u, v) \mapsto (v, \cos(u), \sin(u)), \qquad u \in [0, 2\pi],\ v \in [1, 2].$$

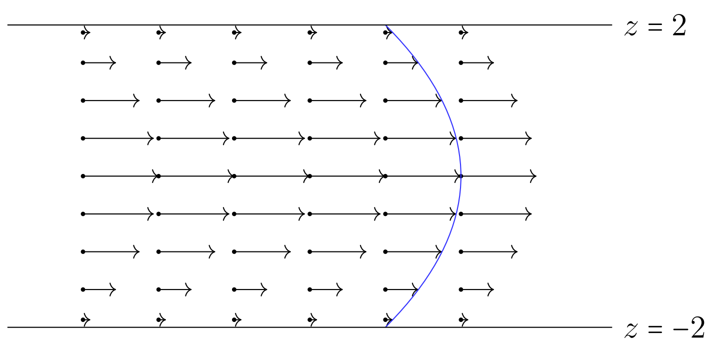
The surface Σ has the same central axis as the cylinder $y^2 + z^2 = 4$, and is contained within the cylinder. Check that $\iint_\Sigma \mathbf{F} \cdot \mathbf{dS} = 0$, and briefly explain.

Solution.

(a) In the xz-plane, we have $y = 0$, and the vector field restricts to the field

$$(x,z) \mapsto \begin{cases} \left(1 - \frac{z^2}{4}, 0\right), & -2 < z < 2 \\ \mathbf{0}, & \text{Otherwise} \end{cases}.$$

The velocity has a parabolic front.



To see the flow in three dimensions, imagine rotating the above picture about the central axis of the cylinder (that is, about the line $z = 0$).

(b) To begin, we need to find the normal vector to S. We have

$$\mathbf{T}_u = (0, \cos(v), \sin(v))$$
$$\mathbf{T}_v = (0, -u\sin(v), u\cos(v))$$

$$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ 0 & \cos(v) & \sin(v) \\ 0 & -u\sin(v) & u\cos(v) \end{pmatrix} = \left(u\cos^2(v) + u\sin^2(v),\ 0-0,\ 0-0\right) = (u, 0, 0).$$

Notice this points along the positive x-direction, as required.

Then, we compute $\mathbf{F}\cdot\mathbf{N}$. Inside the cylinder, we have

$$\mathbf{F}(\sigma(u,v)) = \left(1 - \frac{u^2\cos^2(v) + u^2\sin^2(v)}{4}, 0, 0\right) = \left(1 - \frac{u^2}{4}, 0, 0\right).$$

Therefore,

$$\mathbf{F}(\sigma(u,v))\cdot\mathbf{N}(u,v) = \left(1 - \frac{u^2}{4}\right)u + 0 + 0 = u - \frac{u^3}{4}.$$

Finally, we integrate over the domain of the parametrization, obtaining

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{dS} &= \iint_D \mathbf{F}(\sigma(u,v))\cdot\mathbf{N}(u,v)\,dA \\
&= \int_0^{2\pi}\int_0^2 u - \frac{u^3}{4}\,du dv \\
&= \int_0^{2\pi}\left[\frac{u^2}{2} - \frac{u^4}{16}\right]_0^2 dv \\
&= \int_0^{2\pi} \frac{4}{2} - \frac{16}{16} - 0\,dv \\
&= \int_0^{2\pi} dv \\
&= 2\pi.
\end{aligned}$$

Remark. If we computed the average flux, by dividing the total flux by the area of the disc, we would obtain

$$\text{Average Flux } = \frac{2\pi}{\pi\cdot 2^2} = \frac{1}{2} = \frac{\text{Maximum speed of flow}}{2}.$$

This is a general feature of such flows (called *laminar flows*) — the average flux is equal to one half the maximum speed of the flow (this is not too hard to show by replacing the vector field above by the scaled field $v_{\max}\mathbf{F}$, where $v_{\max}$ is the max speed).

(c) Of course, we expect that this is true, since the flow is everywhere tangent to such a cylinder. To check this formally, let's find the normal vector and check that the dot product of the flow with the normal is equal to zero. For Σ,

$$\begin{aligned}
\mathbf{T}_u &= (0, -\sin(u), \cos(u)) \\
\mathbf{T}_v &= (1, 0, 0)
\end{aligned}$$

$$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ 0 & -\sin(u) & \cos(u) \\ 1 & 0 & 0 \end{pmatrix} = (0, \cos(u), \sin(u)).$$

Therefore,

$$\mathbf{F}(\sigma(u,v)) \cdot \mathbf{N}(u,v) = (*, 0, 0) \cdot (0, \cos(u), \sin(u)) = 0 + 0 + 0 = 0$$

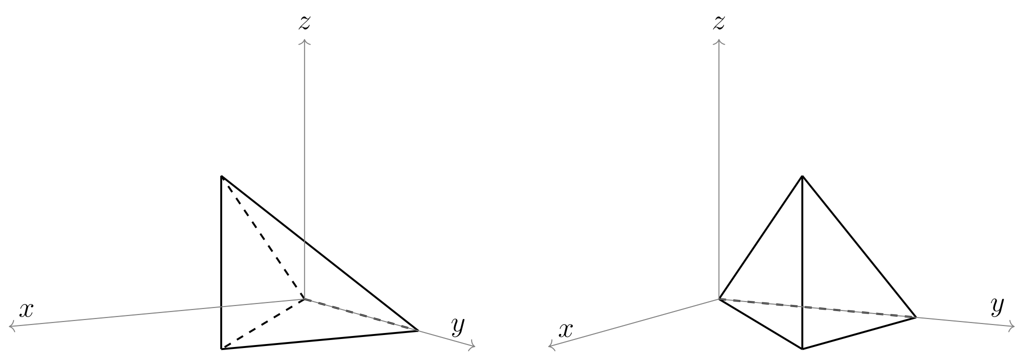
and so

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{dS} = 0.$$

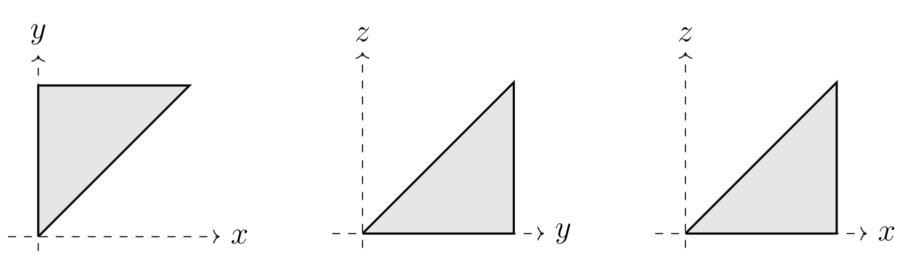
2. Sketch the volume of integration for the iterated integral $\int_0^1 \int_0^y \int_0^x x^2yz \; dz\, dx\, dy$, and express it in the five other possible orders of integration.

(You do not have to evaluate any of the integrals.)

Solution. The volume is a tetrahedron with vertices at $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(1,1,1)$, and bounded by the planes $z = 0$, $x = z$, $x = y$ and $y = 1$.



The projections or shadows to the three coordinate planes look like



Therefore, in the six orders of integration the integral is

$$\int_0^1 \int_0^y \int_0^x x^2yz \; dz\,dx\,dy, \qquad \int_0^1 \int_x^1 \int_0^x x^2yz \; dz\,dy\,dx,$$
$$\int_0^1 \int_0^y \int_z^y x^2yz \; dx\,dz\,dy, \qquad \int_0^1 \int_z^1 \int_z^y x^2yz \; dx\,dy\,dz,$$
$$\int_0^1 \int_0^x \int_x^1 x^2yz dy\,dz\,dx, \qquad \int_0^1 \int_z^1 \int_x^1 x^2yz dy\,dx\,dz.$$

3. Describe the volume of integration, convert to cylindrical coordinates, and evaluate

(a) $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 \; dz\,dy\,dx$,

(b) $\displaystyle\int_{-\sqrt{8}}^{\sqrt{8}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{x^2+y^2-8}^{8-x^2-y^2} 2z \; dz\,dy\,dx$.

Solution.

(a) The volume of integration is bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 3$. Notice that the cone intersects the plane when $3 = \sqrt{x^2+y^2}$.

In cylindrical coordinates, the cone is described by the equality $z = r$, so the triple integral (in one of the possible orders of integration) is (remembering the Jacobian factor of r in cylindrical coordinates)

$$\begin{aligned}\int_0^{2\pi} \int_0^3 \int_0^z r\,dr\,dz\,d\theta &= \int_0^{2\pi} \int_0^3 \left[\frac{r^2}{2}\right]_0^z dz\,d\theta \\ &= \int_0^{2\pi} \int_0^3 \frac{z^2}{2}\,dz\,d\theta \\ &= \int_0^{2\pi} \left[\frac{z^3}{6}\right]_0^3 d\theta \\ &= 2\pi \cdot \frac{27}{6} \\ &= 9\pi.\end{aligned}$$

(b) The volume of integration is bounded below by the paraboloid $z = x^2+y^2-8$ and above by the paraboloid $8-x^2-y^2$. The shadow of the two paraboloids in the xy-plane is a circle of radius $\sqrt{8}$.

In cylindrical coordinates, the two paraboloids are described by $z = r^2 - 8$ (bottom) and $z = 8 - r^2$ (top). The integral is

$$\int_0^{2\pi} \int_0^{\sqrt{8}} \int_{r^2-8}^{8-r^2} 2zr\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{8}} \left[z^2\right]_{r^2-8}^{8-r^2} r\,dr\,d\theta = 0.$$

4. (a) Show that the center of mass of the solid $x^2 + y^2 + z^2 \leq 1,\ z \geq 0$ (the top half of the ball of radius 1) of uniform density δ has Cartesian coordinates $\left(0, 0, \frac{3}{8}\right)$. (Suggestion: Integrate in spherical coordinates.)

(b) Now suppose that the density δ of the solid in part (a) is given by

$$\delta(x, y, z) = 1 - \gamma z$$

for some number $0 \leq \gamma \leq 1$. (Interpretation: the ball is made of lighter material at the top than at the base. The upper bound on γ is made to avoid regions of negative density.)

Find the coordinates of the center of mass as a function of γ. For which γ is the center of mass at the point $\left(0, 0, \frac{1}{3}\right)$?

Solution.

(a) Denote the solid by R, and its density by δ. By symmetry, the center of mass of the top half of the ball must lie on the z-axis. (For instance, if we made a calculation that placed the center of mass in a point with a non-zero x or y-coordinate, we could spin the ball around the z-axis, and repeat the calculation, leading to a contradiction.) To find the z-coordinate, we need to compute

$$\frac{\iiint_R z\delta\, dV}{\iiint_R \delta\, dV}.$$

In the uniform density case, the δ factors out both integrals, and cancels in the fraction, so it is enough to compute

$$\frac{\iiint_R z\, dV}{\iiint_R dV}.$$

To do this, we integrate in spherical coordinates.

The function z converts to $\rho\cos(\phi)$, and R is described by the inequalities

$$0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$$

Finally, we need to remember the Jacobian factor $\rho^2\sin(\phi)$. We have

$$\begin{aligned}
\iiint_R dV &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} \rho^2 \sin(\phi)\, d\phi\, d\rho\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \rho^2 \left[-\cos(\phi)\right]_0^{\pi/2}\, d\rho\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \rho^2\, d\rho\, d\theta \\
&= \int_0^{2\pi}\left[\frac{\rho^3}{3}\right]_0^1 d\theta \\
&= (2\pi)\left(\frac{1}{3} - 0\right) \\
&= \frac{2}{3}\pi.
\end{aligned}$$

This makes sense, since we should be getting half of the volume of a unit sphere, which is equal to $\frac{4}{3}\pi \cdot 1^3$.

Now, the second integral is

$$\begin{aligned}\iiint_R z\,dV &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} (\rho\cos(\phi))\,\rho^2\sin(\phi)\,d\phi\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \rho^3\left[\frac{1}{2}\sin^2(\phi)\right]_0^{\pi/2} d\rho\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left[\frac{\rho^4}{4}\right]_0^1 d\theta \\ &= \frac{1}{2}\cdot 2\pi\cdot\frac{1}{4} \\ &= \frac{\pi}{4}.\end{aligned}$$

Therefore, the z-coordinate of the center of mass is

$$\frac{\iiint_R z\,dV}{\iiint_R dV} = \frac{\frac{\pi}{4}}{\frac{2}{3}\pi} = \frac{3}{8},$$

So that the center of mass is at the point

$$\left(0,\,0,\,\frac{3}{8}\right).$$

(b) Because the new density is a function of z only, the center of mass is again along the z-axis by symmetry. Our task is to compute

$$\frac{\iiint_R z\delta\,dV}{\iiint_R \delta\,dV} = \frac{\iiint_R z-\gamma z^2\,dV}{\iiint_R 1-\gamma z\,dV}.$$

One of the cleaner ways of doing this is to apply the linearity of the integral, and begin by computing

$$\iiint_R dV, \quad \iiint_R z\,dV, \quad \iiint_R z^2\,dV.$$

From part (a), we know that

$$\begin{aligned}\iiint_R dV &= \frac{2}{3}\pi, \\ \iiint_R z\,dV &= \frac{\pi}{4}.\end{aligned}$$

The third integral is

$$\begin{aligned}\iiint_R z^2\,dV &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} (\rho\cos(\phi))^2\,\rho^2\sin(\phi)\,d\phi\,d\rho\,d\theta\\ &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} \rho^4\cos^2(\phi)\sin(\phi)\,d\phi\,d\rho\,d\theta\\ &= \int_0^{2\pi}\int_0^1 \rho^4\left[-\frac{1}{3}\cos^3(\phi)\right]_0^{\pi/2}\,d\rho\,d\theta\\ &= 2\pi\cdot\left[\frac{\rho^5}{5}\right]_0^1\cdot\frac{1}{3}\\ &= \frac{2\pi}{15}.\end{aligned}$$

We then have

$$\begin{aligned}\frac{\iiint_R z-\gamma z^2\,dV}{\iiint_R 1-\gamma z\,dV} &= \frac{\iiint_R z\,dV-\gamma\iiint_R z^2\,dV}{\iiint_R dV-\gamma\iiint_R z\,dV}\\ &= \frac{\pi/4-2\gamma\pi/15}{2\pi/3-\gamma\pi/4}\\ &= \frac{15-8\gamma}{40-15\gamma}.\end{aligned}$$

This is equal to $\frac{1}{3}$ when $3(15-8\gamma)=40-15\gamma$, or, rearranging, $5=9\gamma$, so at

$$\gamma=\frac{5}{9}.$$