MTHE 227 Problem Set 9 Solutions

1 (Cross-Product in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** For this problem, to help distinguish between the cross-products in 2- and 3-space, for vectors

$$\mathbf{v_1} = (x_1, y_1),\ \mathbf{v_2} = (x_2, y_2) \text{ in } \mathbb{R}^2 \quad \text{and} \quad \mathbf{w_1} = (x_1, y_1, z_1),\ \mathbf{w_2} = (x_2, y_2, z_2) \text{ in } \mathbb{R}^3,$$

write

$$\operatorname{cross}_2(\mathbf{v_1}, \mathbf{v_2}) = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \qquad \text{and} \qquad \operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) = \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

Embed $\mathbb{R}^2_{(x,y)}$ into $\mathbb{R}^3_{(x,y,z)}$ by the map $(x, y) \mapsto (x, y, 0)$ (the image being the plane $z = 0$).

(a) Let $\mathbf{v_1}$, $\mathbf{v_2}$ be vectors in $\mathbb{R}^2_{(x,y)}$ and $\mathbf{w_1}$, $\mathbf{w_2}$ their images under the embedding. Check that

$$\operatorname{cross}_2(\mathbf{v_1}, \mathbf{v_2}) = \operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) \cdot \mathbf{e_z}\ .$$

(b) Let $\mathbf{r}\colon t \mapsto (x(t), y(t), 0)$, $t \in [a, b]$ be a parametrized path in $\mathbb{R}^3_{(x,y,z)}$ (thought of as the image of a parametrized path in $\mathbb{R}^2_{(x,y)}$ under the above embedding). Denote the velocity vector at time t by $\mathbf{r}'(t) = (x'(t), y'(t), 0)$. Check that

$$\begin{aligned} \mathbf{n}_+(t) &:= (\, y'(t),\, -x'(t),\, 0\,) = \operatorname{cross}_3(\mathbf{r}', \mathbf{e_z}) \qquad \text{and} \\ \mathbf{n}_-(t) &:= (\, -y'(t),\, x'(t),\, 0\,) = \operatorname{cross}_3(\mathbf{e_z}, \mathbf{r}'). \end{aligned}$$

Solution.

(a) Writing $\mathbf{v}_1 = (x_1, y_1)$, $\mathbf{v}_2 = (x_2, y_2)$, $\mathbf{w}_1 = (x_1, y_1, 0)$, $\mathbf{w}_2 = (x_2, y_2, 0)$, we have (expanding the determinant along the top row)

$$\begin{aligned} \operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) &= \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix} \mathbf{e_x} - \det\begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \mathbf{e_y} + \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mathbf{e_z} \\ &= 0\,\mathbf{e_x} - 0\,\mathbf{e_y} + \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mathbf{e_z}. \end{aligned}$$

Therefore,

$$\operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) \cdot \mathbf{e_z} = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} (\mathbf{e_z} \cdot \mathbf{e_z}) = \operatorname{cross}_2(\mathbf{v_1}, \mathbf{v_2}).$$

The cross product of any pair of vectors lying in a plane will point along the normal direction to the plane. If $\mathbb{R}^2$ is embedded into $\mathbb{R}^3$ as the xy-plane, the cross product of two vectors in the image of $\mathbb{R}^2$ will point along the z-axis; the coefficient of cross_3 along the z-axis is exactly cross_2!

(b) We have

$$\begin{aligned}\operatorname{cross}_3(\mathbf{r}', \mathbf{e_z}) &= \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ x'(t) & y'(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \det\begin{pmatrix} y'(t) & 0 \\ 0 & 1 \end{pmatrix}\mathbf{e_x} - \det\begin{pmatrix} x'(t) & 0 \\ 0 & 1 \end{pmatrix}\mathbf{e_y} + \det\begin{pmatrix} x'(t) & y'(t) \\ 0 & 0 \end{pmatrix}\mathbf{e_z} \\ &= (y'(t), -x'(t), 0) = \mathbf{n}_+(t)\end{aligned}$$

and the other equality follows from the fact that $\operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) = -\operatorname{cross}_3(\mathbf{w_2}, \mathbf{w_1})$ (this follows from a general property of determinants: switching a pair of rows introduces a negative sign).

This gives another way of computing the clockwise and counterclockwise normal vectors to a plane curve.

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Optional Problem (Harder). Embed $\mathbb{R}^2_{(x,y)}$, $\mathbb{R}^2_{(y,z)}$ and $\mathbb{R}^2_{(x,z)}$ into $\mathbb{R}^3_{(x,y,z)}$ as the planes $z = 0$, $x = 0$ and $y = 0$, respectively. Let $\pi_z : \mathbb{R}^3_{(x,y,z)} \to \mathbb{R}^2_{(x,y)}$ be the projection map $(x, y, z) \mapsto (x, y)$, and similarly define π_x, the projection onto $\mathbb{R}^2_{(y,z)}$, and π_y, the projection onto $\mathbb{R}^2_{(x,z)}$.

Let P be a parallelogram in $\mathbb{R}^3$, and denote its images under the above projections by $P_x = \pi_x(P)$, $P_y = \pi_y(P)$ and $P_z = \pi_z(P)$. Show that

$$\operatorname{area}(P) = \sqrt{\operatorname{area}(P_x)^2 + \operatorname{area}(P_y)^2 + \operatorname{area}(P_z)^2}.$$

Conclude, by applying the Cauchy-Schwarz inequality or otherwise, that

$$\operatorname{area}(P) \geq \frac{1}{\sqrt{3}}(\operatorname{area}(P_x)+\operatorname{area}(P_y)+\operatorname{area}(P_z)) = \sqrt{3}\cdot\text{Arithmetic Mean}(\operatorname{area}(P_x), \operatorname{area}(P_y), \operatorname{area}(P_z)).$$

Can you find a P for which equality holds?

2 (Triple Cross Product)**.** Find three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ such that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

(If you are stuck, there is a suggestion at the end of the problem set. But try to find the vectors yourself — there are many possibilities.)

Solution. For instance, we could take $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (1, 0, 0)$ and $\mathbf{w} = (0, 1, 0)$.

Then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, so $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0} \times \mathbf{w} = \mathbf{0}$.

On the other hand, $\mathbf{v} \times \mathbf{w} = (0, 0, 1)$, and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (0, -1, 0) \neq \mathbf{0}$.

The optional problem below characterizes the triples of vectors for which equality holds.

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Optional Problem (Messy). Show the identity

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

by expanding out in coordinates, and conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Conclude that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ if and only if either: $\mathbf{u}$ and $\mathbf{w}$ are both perpendicular to $\mathbf{v}$, or $\mathbf{u} = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{R}$.

Also, conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0} \qquad \text{(the Jacobi identity).}$$

3 (Examples of Centroids of Curves)**.** In lecture, we learned how to compute the coordinates of the center of mass of a curve C in $\mathbb{R}^3$. When C has uniform unit density (that is, $\delta = 1$), the center of mass of C is also called the *centroid*. The coordinates of the centroid of C are then

$$\frac{1}{\int_C ds}\left(\int_C x\,ds,\ \int_C y\,ds,\ \int_C z\,ds\right).$$

A similar expression is true for a curve in $\mathbb{R}^2$, omitting the z-coordinate.

Find the centroids of the following curves in $\mathbb{R}^2$. You may use symmetry arguments to reduce the number of computations you need to do.

(a) The line segment parametrized by $t \mapsto (t,\ mt),\ t \in [0,\ \frac{1}{m}]$, where $m > 0$ is the slope.

(b) The right semicircle $t \mapsto (a\cos(t),\ a\sin(t)),\ t \in [-\frac{\pi}{2},\ \frac{\pi}{2}]$ of radius a centered at the origin.

(c) The circle $t \mapsto (b + a\cos(t),\ a\sin(t)),\ t \in [0, 2\pi]$ of radius a centered at $(b, 0)$, with $b > a$ (feel free to write down the answer without computation if you see it).

(d) The piecewise curve $C = C_1 + C_2 + C_3$, where C_1 is the line segment from $(0, b)$ to (a, b), C_2 the line segment from (a, b) to $(a, -b)$, and C_3 the line segment from $(a, -b)$ to $(0, -b)$, where $a > 0$ and $b > 0$. The curve C is a $a \times 2b$ rectangle, with the left side missing.

(e) Find the integral $\int_C x\,ds$ for the parabola segment $t \mapsto (t,\ t^2),\ t \in [0, 1]$.

Solution. The following solutions include fairly complete computations. It was possible to skip many of these computations by symmetry arguments.

(a) We expect the center of mass to be at $(\frac{1}{2m}, \frac{1}{2})$. The parametrization is $t \mapsto (t,\ mt),\ t \in [0, 1/m]$. The velocity is $\mathbf{v}(t) = (1,\ m)$, hence the speed is $\|\mathbf{v}(t)\| = \sqrt{1 + m^2}$. The three integrals are

$$\int_C ds = \int_0^{1/m} \sqrt{1+m^2}\,dt = \frac{\sqrt{1+m^2}}{m},$$
$$\int_C x\,ds = \int_0^{1/m} t\sqrt{1+m^2}\,dt = \sqrt{1+m^2}\left[\frac{t^2}{2}\right]_0^{1/m} = \frac{\sqrt{1+m^2}}{2m^2},$$
$$\int_C y\,ds = \int_0^{1/m} mt\sqrt{1+m^2}\,dt = \frac{m\sqrt{1+m^2}}{2m^2}.$$

Notice that the length is consistent with Pythagoras' theorem, since the line segment is the hypotenuse of a right triangle with side lengths 1 and $1/m$.

The x-coordinate of the centroid is

$$\frac{\sqrt{1+m^2}}{2m^2}\left(\frac{\sqrt{1+m^2}}{m}\right)^{-1} = \frac{m}{2m^2} = \frac{1}{2m}.$$

Similarly, the y-coordinate is

$$\frac{m\sqrt{1+m^2}}{2m^2}\left(\frac{\sqrt{1+m^2}}{m}\right)^{-1} = \frac{m^2}{2m^2} = \frac{1}{2}.$$

The centroid is located at the point

$$\left(\frac{1}{2m}, \frac{1}{2}\right).$$

as expected.

(b) By symmetry, the centroid should be on the x-axis.

The parametrization is $t \mapsto (a\cos(t),\, a\sin(t)),\ t \in [-\pi/2,\, \pi/2]$. The velocity is $\mathbf{v}(t) = (-a\sin(t),\, a\cos(t))$, the speed is $\|\mathbf{v}(t)\| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = a$. The three integrals are

$$\int_C ds = \int_{-\pi/2}^{\pi/2} a\,dt = a\pi,$$
$$\int_C x\,ds = \int_{-\pi/2}^{\pi/2} a\cos(t)\,a\,dt = a^2\left[\sin(t)\right]_{-\pi/2}^{\pi/2} = a^2(1-(-1)) = 2a^2,$$
$$\int_C y\,ds = \int_{-\pi/2}^{\pi/2} a\sin(t)\,a\,dt = a^2\left[-\cos(t)\right]_{-\pi/2}^{\pi/2} = a^2(-0+0) = 0.$$

The centroid is at the point

$$\left(\frac{2a^2}{\pi a}, 0\right) = \left(\frac{2a}{\pi}, 0\right).$$

(c) The centroid should be at the center of the circle, which is at $(b, 0)$.

The path is $t \mapsto (b+a\cos(t),\, a\sin(t))$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-a\sin(t),\, a\cos(t))$ and the speed is $\|\mathbf{v}(t)\| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = a$. The three integrals are

$$\begin{aligned}
\int_C ds &= \int_0^{2\pi} a\,dt = 2\pi a,\\
\int_C x\,ds &= \int_0^{2\pi} (b + a\cos(t))a\,dt = 2\pi ab + a^2\int_0^{2\pi} \cos(t)\,dt = 2\pi ab,\\
\int_C y\,ds &= \int_0^{2\pi} a\sin(t)a\,dt = 0.
\end{aligned}$$

The centroid is at the point

$$\left(\frac{2\pi ab}{2\pi a},\, 0\right) = (b, 0).$$

(d) Triple the fun! Parametrize the curves C_1, C_2, C_3 as follows:

Curve	Parametrization	$\mathbf{v}(t)$	$\|\mathbf{v}(t)\|$
C_1	$t \mapsto (t, b),\ t \in [0, a]$	$(1, 0)$	1
C_2	$t \mapsto (a, b - t),\ t \in [0, 2b]$	$(0, -1)$	1
C_3	$t \mapsto (a - t, -b),\ t \in [0, a]$	$(-1, 0)$	1

Then,

$$\begin{aligned}
\int_C ds &= \int_0^a dt + \int_0^{2b} dt + \int_0^a dt = 2(a + b),\\
\int_C x\,ds &= \int_0^a t\,dt + \int_0^{2b} a\,dt + \int_0^a (a - t)\,dt\\
&= \left[\frac{t^2}{2}\right]_0^a + 2ab + \left[at - \frac{t^2}{2}\right]_0^a\\
&= \frac{a^2}{2} + 2ab + a^2 - \frac{a^2}{2} = a^2 + 2ab,\\
\int_C y\,ds &= \int_0^a b\,dt + \int_0^{2b} b - t\,dt + \int_0^a -b\,dt\\
&= ab + \left[bt - \frac{t^2}{2}\right]_0^{2b} - ab\\
&= 2b^2 - 2b^2 = 0.
\end{aligned}$$

Therefore, the coordinates of the centroid are

$$\left(\frac{a^2 + 2ab}{2(a + b)}, 0\right).$$

A good shortcut for this part uses the following lemma:

Lemma. *Let $C_1, \ldots, C_n$ be curves in $\mathbb{R}^2$ or $\mathbb{R}^3$. For each $i = 1, \ldots, n$, let M_i denote the mass of the curve C_i, and let $\mathbf{R_i}$ denote the coordinates of its center of mass. Then the coordinates of the center of mass of the union of the curves C_i is equal to*

$$\frac{M_1\mathbf{R_1} + \cdots + M_n\mathbf{R_n}}{M_1 + \cdots + M_n}.$$

Proof. Suppose that C_i are curves in $\mathbb{R}^3$. For each i, we have

$$M_i\mathbf{R_i} = \left(\int_{C_i} x\,\delta(x,y,z)\,ds,\ \int_{C_i} y\,\delta(x,y,z)\,ds,\ \int_{C_i} z\,\delta(x,y,z)\,ds\right) =: \int_{C_i} \mathbf{r}\,\delta(x,y,z)\,ds.$$

Therefore,

$$M_1\mathbf{R_1} + \cdots + M_n\mathbf{R_n} = \int_{C_1} \mathbf{r}\,\delta(x,y,z)\,ds + \cdots + \int_{C_n} \mathbf{r}\,\delta(x,y,z)\,ds = \int_{C} \mathbf{r}\,\delta(x,y,z)\,ds.$$

and

$$\frac{M_1\mathbf{R_1} + \cdots + M_n\mathbf{R_n}}{M_1 + \cdots + M_n}. = \frac{\int_C \mathbf{r}\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds} = \text{Center of Mass}(C).$$

□

For the curves C_1, C_2, C_3, we have (since $\delta = 1$)

$$M_1 = a, \quad M_2 = 2b, \quad M_3 = a;$$
$$\mathbf{R_1} = (a/2, b), \quad \mathbf{R_2} = (a, 0), \quad \mathbf{R_3} = (a/2, -b).$$

Therefore, the centroid is at the point

$$\frac{a(a/2,b) + 2b(a,0) + (a(a/2,-b)}{2(a+b)} = \frac{(a^2/2 + 2ab + a^2/2,\ ab + 0 - ab)}{2(a+b)} = \left(\frac{a^2+2ab}{2(a+b)},\ 0\right).$$

(e) The path is $t \mapsto (t, t^2)$, $t \in [0,1]$. The velocity is $\mathbf{v}(t) = (1, 2t)$ and the speed is $\|\mathbf{v}(t)\| = \sqrt{1+4t^2}$. Therefore,

$$\int_C x\,ds = \int_0^1 t\sqrt{1+4t^2}\,dt.$$

Let $u = 1 + 4t^2$, $du = 8t\,dt$. Making the u-substitution, the integral becomes

$$\frac{1}{8}\int_1^5 \sqrt{u}\,du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{5\sqrt{5}-1}{12}.$$

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Optional Problem (Harder). Find the coordinates of the centroid of the parabola segment in part (e). The standard approach to the integrals involved uses sinh-substitution (!).

4 (Surfaces of Revolution)**.** A surface of revolution is the surface obtained by rotating a plane curve C about a line ℓ (called the axis of rotation) that is coplanar with C.

To obtain a surface according to the definition in lecture, we require that ℓ does not intersect C, except possibly at the endpoints of C. To obtain a smooth surface (except for at most finitely many nonsmooth curves, which do not affect surface area), we require that there exists a parametrization $t \mapsto \mathbf{r}(t) = (x(t), z(t))$, $t \in [a,b]$ of C with $\mathbf{r}'(t) \neq \mathbf{0}$ for all t (with at most finitely many exceptions).

Suppose that C lies in the xz-plane with $x > 0$, ℓ is the z-axis, and fix a parametrization of C as above.

(a) Find the unit vector that is obtained by rotating $\mathbf{e_x}$ counterclockwise by θ radians about the z-axis.

(b) Using the parametrization $t \mapsto \mathbf{r}(t) = (x(t),\, z(t)),\ t \in [a,b]$ of C, parametrize the curve obtained by rotating C counterclockwise by θ radians about the z-axis (it will lie in the plane spanned by $\mathbf{e_z}$ and the vector from part (a)). Your parametrization will involve the functions $x(t)$ and $z(t)$.

(c) Parametrize the surface of revolution of C, taking one of the parameters to be the parameter t of C, and the other parameter to be the angle θ. What do the t- and θ-coordinate curves look like?

(d) Find the tangent vectors $\mathbf{T}_t(t,\theta)$ and $\mathbf{T}_\theta(t,\theta)$ at all points.

(e) Find the normal $\mathbf{N}(t,\theta) = \mathbf{T}_t(t,\theta) \times \mathbf{T}_\theta(t,\theta)$ and its magnitude $\|\mathbf{N}(t,\theta)\|$ at all points.

(f) Show that the surface area of the surface of revolution of C is equal to

$$2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2}\,dt = 2\pi \int_C x\,ds.$$

(g) Conclude that the following theorem holds:

Theorem (Pappus)**.** *The surface area of the surface of revolution of a curve C is equal to the product*

$$\text{arclength}(C) \cdot \text{distance travelled by the centroid of } C.$$

(h) For each of the curves in Problem 3, sketch its surface of revolution about the z-axis and find the surface area using Pappus's theorem.

Solution.

(a) From the geometry, we see that $\cos(\theta)\,\mathbf{e_x} + \sin(\theta)\,\mathbf{e_y}$ is such a vector.

(b) The parametrization of C in the xz-plane may be written

$$t \mapsto x(t)\,\mathbf{e_x} + z(t)\,\mathbf{e_z},\ t \in [a,b].$$

To parametrize the rotated curve, replace $\mathbf{e_x}$ by the vector found in part (a), obtaining the parametrization

$$t \mapsto (\,x(t)\cos(\theta),\, x(t)\sin(\theta),\, z(t)\,)\,,\ t \in [a,b].$$

(c) Allowing θ to be arbitrary in the parametrization from part (b) yields a parametrization of the surface of revolution:

$$(t,\theta) \mapsto (\,x(t)\cos(\theta),\, x(t)\sin(\theta),\, z(t)\,)\,, \quad t \in [a,b],\ \theta \in [0,2\pi].$$

(d) We compute

$$\mathbf{T}_t(t,\theta) = (x'(t)\cos(\theta),\, x'(t)\sin(\theta),\, z'(t))\,,$$
$$\mathbf{T}_\theta(t,\theta) = (-x(t)\sin(\theta),\, x(t)\cos(\theta),\, 0)\,.$$

(e) Computing the cross product,

$$\begin{aligned}\mathbf{T}_t \times \mathbf{T}_\theta &= \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ x'(t)\cos(\theta) & x'(t)\sin(\theta) & z'(t) \\ -x(t)\sin(\theta) & x(t)\cos(\theta) & 0 \end{pmatrix} \\ &= \left(0 - z'(t)x(t)\cos(\theta),\, -(0 + z'(t)x(t)\sin(\theta)),\, x(t)x'(t)\cos^2(\theta) + x(t)x'(t)\sin^2(\theta)\right) \\ &= (-x(t)z'(t)\cos(\theta),\; -x(t)z'(t)\sin(\theta),\; x(t)x'(t))\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{N}(t,\theta)\|^2 &= x^2(t)z'(t)^2\cos^2(\theta) + x^2(t)z'(t)^2\sin^2(\theta) + x^2(t)x'(t)^2 \\ &= x^2(t)\left(x'(t)^2 + z'(t)^2\right)\end{aligned}$$

and

$$\|\mathbf{N}(t,\theta)\| = x(t)\sqrt{x'(t)^2 + z'(t)^2}.$$

(f) The surface area of the surface of revolution is therefore equal to

$$\iint_S dS = \int_a^b \int_0^{2\pi} \|\mathbf{N}(t,\theta)\|\, d\theta dt = 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2} = 2\pi \int_C x\, ds.$$

(g) The centroid of C travels around the z-axis in a circle with radius equal to its x-coordinate. Therefore,

$$\text{arclength}(C)\cdot\text{distance travelled by the centroid of } C = \left(\int_C ds\right)\left(2\pi \frac{\int_C x\, ds}{\int_C ds}\right) = 2\pi \int_C x\, ds,$$

and the expression on the right is equal to the surface area of the surface of revolution of C from part (f).

It is worth noticing that if the centroid of C is not clear from symmetry, it is computationally simpler to find $\int_C x\, ds$. The advantage of the above formulation of Pappus' theorem is that often centroids of symmetric shapes are simple to find without computation (we have seen a few examples in Problem 3!).

(h) (A) The surface of revolution is a cone:

The coordinates of the centroid are $\left(\frac{1}{2m}, \frac{1}{2}\right)$. The arclength of the line segment is $\frac{\sqrt{1+m^2}}{m}$. Therefore, the surface area of the cone is

$$\frac{\sqrt{1+m^2}}{m} 2\pi \frac{1}{2m} = \frac{\pi}{m}\sqrt{1+\frac{1}{m^2}}.$$

(B) The surface of revolution is a sphere of radius a:

The coordinates of the centroid are $\left(\frac{2a}{\pi}, 0\right)$. The arclength of a semicircle of radius a is πa. Therefore, the surface area of the sphere is

$$\pi a \cdot 2\pi \frac{2a}{\pi} = 4\pi a^2,$$

agreeing with what we found before!

(C) The surface of revolution is a torus with radii a and b:

The coordinates of the centroid are $(b, 0)$. The arclength of the generating circle is $2\pi a$. Therefore, the surface area of the torus is

$$2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

(D) The surface of revolution is a cylinder with caps on top and bottom:

The coordinates of the centroid are $\left(\dfrac{a^2+2ab}{2(a+b)}, 0\right)$, and the arclength is $2(a+b)$, so the surface area of the surface of revolution is

$$2\pi a^2 + 4\pi ab.$$

Notice that this is the sum of the surface area of the top and bottom disks, together with the side of the cylinder.

(E) The surface of revolution is a paraboloid.

In this instance, the coordinates of the centroid are not easy to compute, but we have found that $\int_C x\,ds = \dfrac{5\sqrt{5}-1}{12}$, so the surface area is

$$2\pi\frac{5\sqrt{5}-1}{12} = \pi\frac{5\sqrt{5}-1}{6}.$$