

MTHE 227 PROBLEM SET 9 SOLUTIONS

1 (Cross-Product in \mathbb{R}^2 and \mathbb{R}^3). For this problem, to help distinguish between the cross-products in 2- and 3-space, for vectors

$$\mathbf{v}_1 = (x_1, y_1), \mathbf{v}_2 = (x_2, y_2) \text{ in } \mathbb{R}^2 \quad \text{and} \quad \mathbf{w}_1 = (x_1, y_1, z_1), \mathbf{w}_2 = (x_2, y_2, z_2) \text{ in } \mathbb{R}^3,$$

write

$$\text{cross}_2(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \quad \text{and} \quad \text{cross}_3(\mathbf{w}_1, \mathbf{w}_2) = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

Embed $\mathbb{R}_{(x,y)}^2$ into $\mathbb{R}_{(x,y,z)}^3$ by the map $(x, y) \mapsto (x, y, 0)$ (the image being the plane $z = 0$).

- (a) Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}_{(x,y)}^2$ and $\mathbf{w}_1, \mathbf{w}_2$ their images under the embedding. Check that

$$\text{cross}_2(\mathbf{v}_1, \mathbf{v}_2) = \text{cross}_3(\mathbf{w}_1, \mathbf{w}_2) \cdot \mathbf{e}_z .$$

- (b) Let $\mathbf{r}: t \mapsto (x(t), y(t), 0)$, $t \in [a, b]$ be a parametrized path in $\mathbb{R}_{(x,y,z)}^3$ (thought of as the image of a parametrized path in $\mathbb{R}_{(x,y)}^2$ under the above embedding). Denote the velocity vector at time t by $\mathbf{r}'(t) = (x'(t), y'(t), 0)$. Check that

$$\begin{aligned} \mathbf{n}_+(t) &:= (y'(t), -x'(t), 0) = \text{cross}_3(\mathbf{r}', \mathbf{e}_z) \quad \text{and} \\ \mathbf{n}_-(t) &:= (-y'(t), x'(t), 0) = \text{cross}_3(\mathbf{e}_z, \mathbf{r}'). \end{aligned}$$

Solution.

- (a) Writing $\mathbf{v}_1 = (x_1, y_1)$, $\mathbf{v}_2 = (x_2, y_2)$, $\mathbf{w}_1 = (x_1, y_1, 0)$, $\mathbf{w}_2 = (x_2, y_2, 0)$, we have (expanding the determinant along the top row)

$$\begin{aligned} \text{cross}_3(\mathbf{w}_1, \mathbf{w}_2) &= \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix} \mathbf{e}_x - \det \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \mathbf{e}_y + \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mathbf{e}_z \\ &= 0 \mathbf{e}_x - 0 \mathbf{e}_y + \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mathbf{e}_z. \end{aligned}$$

Therefore,

$$\text{cross}_3(\mathbf{w}_1, \mathbf{w}_2) \cdot \mathbf{e}_z = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} (\mathbf{e}_z \cdot \mathbf{e}_z) = \text{cross}_2(\mathbf{v}_1, \mathbf{v}_2).$$

The cross product of any pair of vectors lying in a plane will point along the normal direction to the plane. If \mathbb{R}^2 is embedded into \mathbb{R}^3 as the xy -plane, the cross product of two vectors in the image of \mathbb{R}^2 will point along the z -axis; the coefficient of cross_3 along the z -axis is exactly cross_2 !

(b) We have

$$\begin{aligned}\text{cross}_3(\mathbf{r}', \mathbf{e}_z) &= \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x'(t) & y'(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} y'(t) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{e}_x - \det \begin{pmatrix} x'(t) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{e}_y + \det \begin{pmatrix} x'(t) & y'(t) \\ 0 & 0 \end{pmatrix} \mathbf{e}_z \\ &= (y'(t), -x'(t), 0) = \mathbf{n}_+(t)\end{aligned}$$

and the other equality follows from the fact that $\text{cross}_3(\mathbf{w}_1, \mathbf{w}_2) = -\text{cross}_3(\mathbf{w}_2, \mathbf{w}_1)$ (this follows from a general property of determinants: switching a pair of rows introduces a negative sign).

This gives another way of computing the clockwise and counterclockwise normal vectors to a plane curve.

Optional Problem (Harder). Embed $\mathbb{R}_{(x,y)}^2$, $\mathbb{R}_{(y,z)}^2$ and $\mathbb{R}_{(x,z)}^2$ into $\mathbb{R}_{(x,y,z)}^3$ as the planes $z = 0$, $x = 0$ and $y = 0$, respectively. Let $\pi_z: \mathbb{R}_{(x,y,z)}^3 \rightarrow \mathbb{R}_{(x,y)}^2$ be the projection map $(x, y, z) \mapsto (x, y)$, and similarly define π_x , the projection onto $\mathbb{R}_{(y,z)}^2$, and π_y , the projection onto $\mathbb{R}_{(x,z)}^2$.

Let P be a parallelogram in \mathbb{R}^3 , and denote its images under the above projections by $P_x = \pi_x(P)$, $P_y = \pi_y(P)$ and $P_z = \pi_z(P)$. Show that

$$\text{area}(P) = \sqrt{\text{area}(P_x)^2 + \text{area}(P_y)^2 + \text{area}(P_z)^2}.$$

Conclude, by applying the Cauchy-Schwarz inequality or otherwise, that

$$\text{area}(P) \geq \frac{1}{\sqrt{3}}(\text{area}(P_x) + \text{area}(P_y) + \text{area}(P_z)) = \sqrt{3} \cdot \text{Arithmetic Mean}(\text{area}(P_x), \text{area}(P_y), \text{area}(P_z)).$$

Can you find a P for which equality holds?

2 (Triple Cross Product). Find three vectors \mathbf{u} , \mathbf{v} , \mathbf{w} in \mathbb{R}^3 such that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

(If you are stuck, there is a suggestion at the end of the problem set. But try to find the vectors yourself — there are many possibilities.)

Solution. For instance, we could take $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (1, 0, 0)$ and $\mathbf{w} = (0, 1, 0)$.

Then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, so $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0} \times \mathbf{w} = \mathbf{0}$.

On the other hand, $\mathbf{v} \times \mathbf{w} = (0, 0, 1)$, and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (0, -1, 0) \neq \mathbf{0}$.

The optional problem below characterizes the triples of vectors for which equality holds.

Optional Problem (Messy). Show the identity

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

by expanding out in coordinates, and conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Conclude that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if either: \mathbf{u} and \mathbf{w} are both perpendicular to \mathbf{v} , or $\mathbf{u} = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{R}$.

Also, conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0} \quad (\text{the Jacobi identity}).$$

3 (Examples of Centroids of Curves). In lecture, we learned how to compute the coordinates of the center of mass of a curve C in \mathbb{R}^3 . When C has uniform unit density (that is, $\delta = 1$), the center of mass of C is also called the *centroid*. The coordinates of the centroid of C are then

$$\frac{1}{\int_C ds} \left(\int_C x ds, \int_C y ds, \int_C z ds \right).$$

A similar expression is true for a curve in \mathbb{R}^2 , omitting the z -coordinate.

Find the centroids of the following curves in \mathbb{R}^2 . You may use symmetry arguments to reduce the number of computations you need to do.

- The line segment parametrized by $t \mapsto (t, mt)$, $t \in [0, \frac{1}{m}]$, where $m > 0$ is the slope.
- The right semicircle $t \mapsto (a \cos(t), a \sin(t))$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ of radius a centered at the origin.
- The circle $t \mapsto (b + a \cos(t), a \sin(t))$, $t \in [0, 2\pi]$ of radius a centered at $(b, 0)$, with $b > a$ (feel free to write down the answer without computation if you see it).
- The piecewise curve $C = C_1 + C_2 + C_3$, where C_1 is the line segment from $(0, b)$ to (a, b) , C_2 the line segment from (a, b) to $(a, -b)$, and C_3 the line segment from $(a, -b)$ to $(0, -b)$, where $a > 0$ and $b > 0$. The curve C is a $a \times 2b$ rectangle, with the left side missing.
- Find the integral $\int_C x ds$ for the parabola segment $t \mapsto (t, t^2)$, $t \in [0, 1]$.

Solution. The following solutions include fairly complete computations. It was possible to skip many of these computations by symmetry arguments.

- We expect the center of mass to be at $(\frac{1}{2m}, \frac{1}{2})$. The parametrization is $t \mapsto (t, mt)$, $t \in [0, 1/m]$. The velocity is $\mathbf{v}(t) = (1, m)$, hence the speed is $\|\mathbf{v}(t)\| = \sqrt{1 + m^2}$. The three

integrals are

$$\begin{aligned}\int_C ds &= \int_0^{1/m} \sqrt{1+m^2} dt = \frac{\sqrt{1+m^2}}{m}, \\ \int_C x ds &= \int_0^{1/m} t \sqrt{1+m^2} dt = \sqrt{1+m^2} \left[\frac{t^2}{2} \right]_0^{1/m} = \frac{\sqrt{1+m^2}}{2m^2}, \\ \int_C y ds &= \int_0^{1/m} mt \sqrt{1+m^2} dt = \frac{m\sqrt{1+m^2}}{2m^2}.\end{aligned}$$

Notice that the length is consistent with Pythagoras' theorem, since the line segment is the hypotenuse of a right triangle with side lengths 1 and $1/m$.

The x -coordinate of the centroid is

$$\frac{\sqrt{1+m^2}}{2m^2} \left(\frac{\sqrt{1+m^2}}{m} \right)^{-1} = \frac{m}{2m^2} = \frac{1}{2m}.$$

Similarly, the y -coordinate is

$$\frac{m\sqrt{1+m^2}}{2m^2} \left(\frac{\sqrt{1+m^2}}{m} \right)^{-1} = \frac{m^2}{2m^2} = \frac{1}{2}.$$

The centroid is located at the point

$$\left(\frac{1}{2m}, \frac{1}{2} \right).$$

as expected.

- (b) By symmetry, the centroid should be on the x -axis.

The parametrization is $t \mapsto (a \cos(t), a \sin(t))$, $t \in [-\pi/2, \pi/2]$. The velocity is $\mathbf{v}(t) = (-a \sin(t), a \cos(t))$, the speed is $\|\mathbf{v}(t)\| = \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)} = a$. The three integrals are

$$\begin{aligned}\int_C ds &= \int_{-\pi/2}^{\pi/2} a dt = a\pi, \\ \int_C x ds &= \int_{-\pi/2}^{\pi/2} a \cos(t) a dt = a^2 [\sin(t)]_{-\pi/2}^{\pi/2} = a^2(1 - (-1)) = 2a^2, \\ \int_C y ds &= \int_{-\pi/2}^{\pi/2} a \sin(t) a dt = a^2 [-\cos(t)]_{-\pi/2}^{\pi/2} = a^2(-0 + 0) = 0.\end{aligned}$$

The centroid is at the point

$$\left(\frac{2a^2}{\pi a}, 0 \right) = \left(\frac{2a}{\pi}, 0 \right).$$

- (c) The centroid should be at the center of the circle, which is at $(b, 0)$.

The path is $t \mapsto (b+a \cos(t), a \sin(t))$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-a \sin(t), a \cos(t))$ and the speed is $\|\mathbf{v}(t)\| = \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)} = a$. The three integrals are

$$\begin{aligned}\int_C ds &= \int_0^{2\pi} a dt = 2\pi a, \\ \int_C x ds &= \int_0^{2\pi} (b + a \cos(t))a dt = 2\pi ab + a^2 \int_0^{2\pi} \cos(t) dt = 2\pi ab, \\ \int_C y ds &= \int_0^{2\pi} a \sin(t)a dt = 0.\end{aligned}$$

The centroid is at the point

$$\left(\frac{2\pi ab}{2\pi a}, 0 \right) = (b, 0).$$

(d) Triple the fun! Parametrize the curves C_1, C_2, C_3 as follows:

Curve	Parametrization	$\mathbf{v}(t)$	$\ \mathbf{v}(t)\ $
C_1	$t \mapsto (t, b), t \in [0, a]$	$(1, 0)$	1
C_2	$t \mapsto (a, b - t), t \in [0, 2b]$	$(0, -1)$	1
C_3	$t \mapsto (a - t, -b), t \in [0, a]$	$(-1, 0)$	1

Then,

$$\begin{aligned}\int_C ds &= \int_0^a dt + \int_0^{2b} dt + \int_0^a dt = 2(a + b), \\ \int_C x ds &= \int_0^a t dt + \int_0^{2b} a dt + \int_0^a (a - t) dt \\ &= \left[\frac{t^2}{2} \right]_0^a + 2ab + \left[at - \frac{t^2}{2} \right]_0^a \\ &= \frac{a^2}{2} + 2ab + a^2 - \frac{a^2}{2} = a^2 + 2ab, \\ \int_C y ds &= \int_0^a b dt + \int_0^{2b} b - t dt + \int_0^a -b dt \\ &= ab + \left[bt - \frac{t^2}{2} \right]_0^{2b} - ab \\ &= 2b^2 - 2b^2 = 0.\end{aligned}$$

Therefore, the coordinates of the centroid are

$$\left(\frac{a^2 + 2ab}{2(a + b)}, 0 \right).$$

A good shortcut for this part uses the following lemma:

Lemma. Let C_1, \dots, C_n be curves in \mathbb{R}^2 or \mathbb{R}^3 . For each $i = 1, \dots, n$, let M_i denote the mass of the curve C_i , and let \mathbf{R}_i denote the coordinates of its center of mass. Then the coordinates of the center of mass of the union of the curves C_i is equal to

$$\frac{M_1 \mathbf{R}_1 + \dots + M_n \mathbf{R}_n}{M_1 + \dots + M_n}.$$

Proof. Suppose that C_i are curves in \mathbb{R}^3 . For each i , we have

$$M_i \mathbf{R}_i = \left(\int_{C_i} x \delta(x, y, z) ds, \int_{C_i} y \delta(x, y, z) ds, \int_{C_i} z \delta(x, y, z) ds \right) =: \int_{C_i} \mathbf{r} \delta(x, y, z) ds.$$

Therefore,

$$M_1 \mathbf{R}_1 + \cdots + M_n \mathbf{R}_n = \int_{C_1} \mathbf{r} \delta(x, y, z) ds + \cdots + \int_{C_n} \mathbf{r} \delta(x, y, z) ds = \int_C \mathbf{r} \delta(x, y, z) ds.$$

and

$$\frac{M_1 \mathbf{R}_1 + \cdots + M_n \mathbf{R}_n}{M_1 + \cdots + M_n} = \frac{\int_C \mathbf{r} \delta(x, y, z) ds}{\int_C \delta(x, y, z) ds} = \text{Center of Mass}(C).$$

□

For the curves C_1, C_2, C_3 , we have (since $\delta = 1$)

$$\begin{aligned} M_1 &= a, & M_2 &= 2b, & M_3 &= a; \\ \mathbf{R}_1 &= (a/2, b), & \mathbf{R}_2 &= (a, 0), & \mathbf{R}_3 &= (a/2, -b). \end{aligned}$$

Therefore, the centroid is at the point

$$\frac{a(a/2, b) + 2b(a, 0) + (a(a/2, -b))}{2(a+b)} = \frac{(a^2/2 + 2ab + a^2/2, ab + 0 - ab)}{2(a+b)} = \left(\frac{a^2 + 2ab}{2(a+b)}, 0 \right).$$

- (e) The path is $t \mapsto (t, t^2)$, $t \in [0, 1]$. The velocity is $\mathbf{v}(t) = (1, 2t)$ and the speed is $\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2}$. Therefore,

$$\int_C x ds = \int_0^1 t \sqrt{1 + 4t^2} dt.$$

Let $u = 1 + 4t^2$, $du = 8t dt$. Making the u -substitution, the integral becomes

$$\frac{1}{8} \int_1^5 \sqrt{u} du = \frac{1}{8} \left[\frac{2}{3} u^{3/2} \right]_1^5 = \frac{5\sqrt{5} - 1}{12}.$$

Optional Problem (Harder). Find the coordinates of the centroid of the parabola segment in part (e). The standard approach to the integrals involved uses sinh-substitution (!).

4 (Surfaces of Revolution). A surface of revolution is the surface obtained by rotating a plane curve C about a line ℓ (called the axis of rotation) that is coplanar with C .

To obtain a surface according to the definition in lecture, we require that ℓ does not intersect C , except possibly at the endpoints of C . To obtain a smooth surface (except for at most finitely many nonsmooth curves, which do not affect surface area), we require that there exists a parametrization $t \mapsto \mathbf{r}(t) = (x(t), z(t))$, $t \in [a, b]$ of C with $\mathbf{r}'(t) \neq \mathbf{0}$ for all t (with at most finitely many exceptions).

Suppose that C lies in the xz -plane with $x > 0$, ℓ is the z -axis, and fix a parametrization of C as above.

- (a) Find the unit vector that is obtained by rotating \mathbf{e}_x counterclockwise by θ radians about the z -axis.
- (b) Using the parametrization $t \mapsto \mathbf{r}(t) = (x(t), z(t))$, $t \in [a, b]$ of C , parametrize the curve obtained by rotating C counterclockwise by θ radians about the z -axis (it will lie in the plane spanned by \mathbf{e}_z and the vector from part (a)). Your parametrization will involve the functions $x(t)$ and $z(t)$.
- (c) Parametrize the surface of revolution of C , taking one of the parameters to be the parameter t of C , and the other parameter to be the angle θ . What do the t - and θ -coordinate curves look like?
- (d) Find the tangent vectors $\mathbf{T}_t(t, \theta)$ and $\mathbf{T}_\theta(t, \theta)$ at all points.
- (e) Find the normal $\mathbf{N}(t, \theta) = \mathbf{T}_t(t, \theta) \times \mathbf{T}_\theta(t, \theta)$ and its magnitude $\|\mathbf{N}(t, \theta)\|$ at all points.
- (f) Show that the surface area of the surface of revolution of C is equal to

$$2\pi \int_a^b x(t) \sqrt{x'(t)^2 + z'(t)^2} dt = 2\pi \int_C x ds.$$

- (g) Conclude that the following theorem holds:

Theorem (Pappus). *The surface area of the surface of revolution of a curve C is equal to the product*

$$\text{arclength}(C) \cdot \text{distance travelled by the centroid of } C.$$

- (h) For each of the curves in Problem 3, sketch its surface of revolution about the z -axis and find the surface area using Pappus's theorem.

Solution.

- (a) From the geometry, we see that $\cos(\theta) \mathbf{e}_x + \sin(\theta) \mathbf{e}_y$ is such a vector.
- (b) The parametrization of C in the xz -plane may be written

$$t \mapsto x(t) \mathbf{e}_x + z(t) \mathbf{e}_z, \quad t \in [a, b].$$

To parametrize the rotated curve, replace \mathbf{e}_x by the vector found in part (a), obtaining the parametrization

$$t \mapsto (x(t) \cos(\theta), x(t) \sin(\theta), z(t)), \quad t \in [a, b].$$

- (c) Allowing θ to be arbitrary in the parametrization from part (b) yields a parametrization of the surface of revolution:

$$(t, \theta) \mapsto (x(t) \cos(\theta), x(t) \sin(\theta), z(t)), \quad t \in [a, b], \quad \theta \in [0, 2\pi].$$

(d) We compute

$$\begin{aligned}\mathbf{T}_t(t, \theta) &= (x'(t) \cos(\theta), x'(t) \sin(\theta), z'(t)), \\ \mathbf{T}_\theta(t, \theta) &= (-x(t) \sin(\theta), x(t) \cos(\theta), 0).\end{aligned}$$

(e) Computing the cross product,

$$\begin{aligned}\mathbf{T}_t \times \mathbf{T}_\theta &= \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x'(t) \cos(\theta) & x'(t) \sin(\theta) & z'(t) \\ -x(t) \sin(\theta) & x(t) \cos(\theta) & 0 \end{pmatrix} \\ &= (0 - z'(t)x(t) \cos(\theta), -(0 + z'(t)x(t) \sin(\theta)), x(t)x'(t) \cos^2(\theta) + x(t)x'(t) \sin^2(\theta)) \\ &= (-x(t)z'(t) \cos(\theta), -x(t)z'(t) \sin(\theta), x(t)x'(t)).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{N}(t, \theta)\|^2 &= x^2(t)z'(t)^2 \cos^2(\theta) + x^2(t)z'(t)^2 \sin^2(\theta) + x^2(t)x'(t)^2 \\ &= x^2(t)(x'(t)^2 + z'(t)^2)\end{aligned}$$

and

$$\|\mathbf{N}(t, \theta)\| = x(t)\sqrt{x'(t)^2 + z'(t)^2}.$$

(f) The surface area of the surface of revolution is therefore equal to

$$\iint_S dS = \int_a^b \int_0^{2\pi} \|\mathbf{N}(t, \theta)\| d\theta dt = 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2} dt = 2\pi \int_C x ds.$$

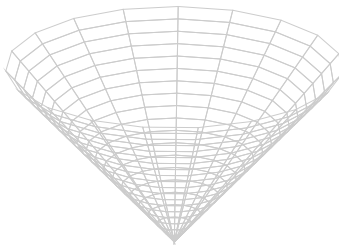
(g) The centroid of C travels around the z -axis in a circle with radius equal to its x -coordinate. Therefore,

$$\text{arlength}(C) \cdot \text{distance travelled by the centroid of } C = \left(\int_C ds \right) \left(2\pi \frac{\int_C x ds}{\int_C ds} \right) = 2\pi \int_C x ds,$$

and the expression on the right is equal to the surface area of the surface of revolution of C from part (f).

It is worth noticing that if the centroid of C is not clear from symmetry, it is computationally simpler to find $\int_C x ds$. The advantage of the above formulation of Pappus' theorem is that often centroids of symmetric shapes are simple to find without computation (we have seen a few examples in Problem 3!).

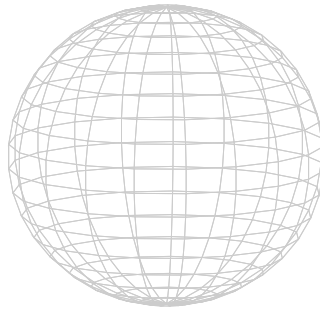
(h) (A) The surface of revolution is a cone:



The coordinates of the centroid are $\left(\frac{1}{2m}, \frac{1}{2}\right)$. The arclength of the line segment is $\frac{\sqrt{1+m^2}}{m}$. Therefore, the surface area of the cone is

$$\frac{\sqrt{1+m^2}}{m} 2\pi \frac{1}{2m} = \frac{\pi}{m} \sqrt{1 + \frac{1}{m^2}}.$$

(B) The surface of revolution is a sphere of radius a :

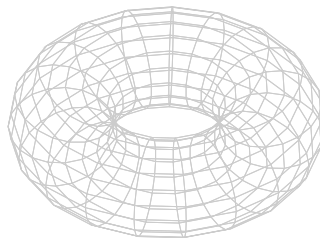


The coordinates of the centroid are $\left(\frac{2a}{\pi}, 0\right)$. The arclength of a semicircle of radius a is πa . Therefore, the surface area of the sphere is

$$\pi a \cdot 2\pi \frac{2a}{\pi} = 4\pi a^2,$$

agreeing with what we found before!

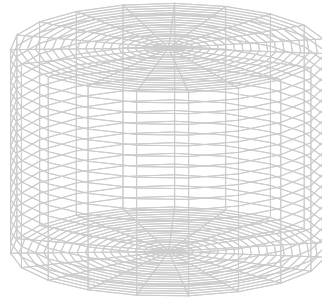
(C) The surface of revolution is a torus with radii a and b :



The coordinates of the centroid are $(b, 0)$. The arclength of the generating circle is $2\pi a$. Therefore, the surface area of the torus is

$$2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

(D) The surface of revolution is a cylinder with caps on top and bottom:

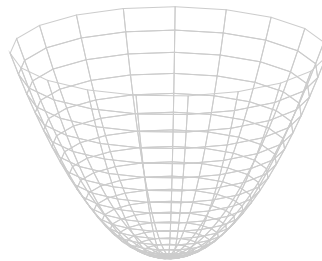


The coordinates of the centroid are $\left(\frac{a^2 + 2ab}{2(a + b)}, 0\right)$, and the arclength is $2(a + b)$, so the surface area of the surface of revolution is

$$2\pi a^2 + 4\pi ab.$$

Notice that this is the sum of the surface area of the top and bottom disks, together with the side of the cylinder.

(E) The surface of revolution is a paraboloid.



In this instance, the coordinates of the centroid are not easy to compute, but we have found that $\int_C x ds = \frac{5\sqrt{5} - 1}{12}$, so the surface area is

$$2\pi \frac{5\sqrt{5} - 1}{12} = \pi \frac{5\sqrt{5} - 1}{6}.$$