MTHE 227 Problem Set 8 Solutions

1 (Path-Connected and Simply-Connected)**.** Which of the following spaces are path-connected? Which are simply-connected? (For cases that are not path-connected, draw two points that cannot be joined by a path. For cases that are path- but not simply-connected, draw a simple closed curve (i.e. a loop) that cannot be continuously deformed to a point while staying in the region. For cases that are simply-connected, it is enough to state this (you do not have to justify it).)

(a) $\mathbb{R}^2$ with the circle $x^2 + y^2 = 1$ removed.

(b) $\mathbb{R}^3$ with the circle $x^2 + y^2 = 1,\ z = 0$ removed.

(c) The annulus $\{(x, y) : 1 < x^2 + y^2 < 2\}$ in $\mathbb{R}^2$.

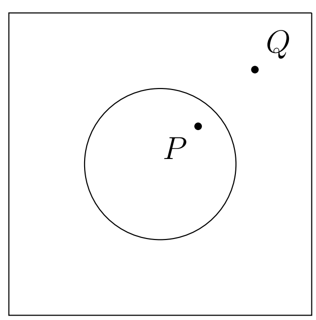
(d) $\mathbb{R}^3$ with a point removed.

(e) $\mathbb{R}^3$ with a line removed.

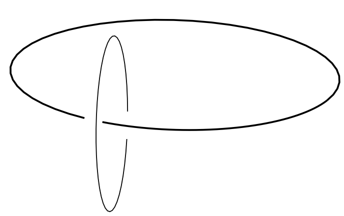
(f) $\mathbb{R}^3$ with the helix $(\cos t, \sin t, t),\ t \in [0, 4\pi]$ removed.

Solution.

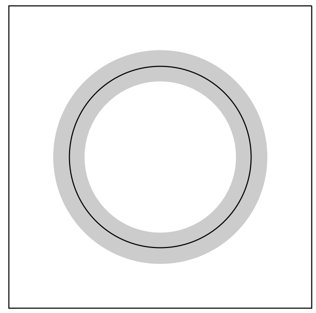
(a) Not path-connected. Take one point inside the circle, and one point outside.



(b) Path-connected, but not simply-connected. Take a loop around the missing circle.

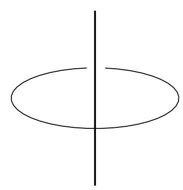


(c) Path-connected, but not simply-connected. Take a loop enclosing the missing inside disk.



(d) Simply-connected.

(e) Path-connected, but not simply-connected. Take a loop around the missing line.



(In the above picture, the line extends infinitely in both directions!)

(f) Simply-connected.

2 (Curl Test)**.** In lecture, we have shown the following theorem:

Theorem (Curl Test)**.** *Let* $\mathbf{F}(x,y) = (\ F_1(x,y),\ F_2(x,y)\ )$ *be a vector field defined in a simply-connected region* X*. If*

$$\operatorname{curl}\mathbf{F} := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

at every point of X*, then* $\mathbf{F}$ *is conservative.*

Conversely, let $\mathbf{G}(x,y) = (\ G_1(x,y),\ G_2(x,y)\ )$ *be a vector field defined in* any *region* X *(not necessarily simply-connected). If* $\operatorname{curl}\mathbf{G}(x,y) \neq 0$ *for some point* (x,y) *in* X*, then* $\mathbf{G}$ *is not conservative.*

Applying the curl test, show that the following vector fields defined on $\mathbb{R}^2$ are not conservative.

(a) $(\, x\sin(y^2),\, y\,\sin(x^2)\,)$.

(b) $(\, 2x + 3y^2 + 5x^3,\, 5y + 3x^2 + 2y^3\,)$.

On the other hand, show that the following vector fields defined are conservative, again applying the curl test:

(c) $\left(\ln y + \dfrac{y}{x},\, \ln x + \dfrac{x}{y} \right)$ on the region with $x > 0,\ y > 0$ (the first quadrant).

(d) $(\,(1 + xy)\, e^{xy},\, x^2 e^{xy}\,)$ on $\mathbb{R}^2$.

Remark. We found potential functions for the vector fields of parts (c) and (d) in Problem Set 4: possibilities are $y \ln x + x \ln y$ for part (c) and $x\, e^{xy}$ for part (d).

Solution.

(a) We have

$$\frac{\partial F_2}{\partial x} = y\cos(x^2)2x, \qquad \frac{\partial F_1}{\partial y} = x\cos(y^2)2y.$$

So

$$\operatorname{curl} \mathbf{F} = 2xy(\cos(x^2) - \cos(y^2)).$$

At $(\sqrt{\pi}, \sqrt{\pi/2})$, the curl is equal to

$$\sqrt{2}\pi(\cos(\pi) - \cos(\pi/2)) = \sqrt{2}\pi(-1 - 0) \neq 0,$$

so the field is not conservative.

(b) We have

$$\frac{\partial F_2}{\partial x} = 6x, \qquad \frac{\partial F_1}{\partial y} = 6y.$$

So

$$\operatorname{curl} \mathbf{F} = 6(x - y).$$

At $(x, y) = (1, 0)$, the curl is nonzero, so the field is not conservative.

(c) **Note:** This problem was originally not stated correctly, since the components of $\mathbf{F}$ aren't defined on the lines $x = 0$ and $y = 0$ (and cannot be extended even continuously to these lines — F_1 approaches infinity as x approaches 0, and F_2 approaches infinity as y approaches 0!). Moreover, the functions $\ln(x)$ and $\ln(y)$ are not defined for negative x and y. So, the problem should be stated for the first quadrant $x > 0$, $y > 0$.

For points in the first quadrant, we have

$$\frac{\partial F_2}{\partial x} = \frac{1}{x} + \frac{1}{y}, \qquad \frac{\partial F_1}{\partial y} = \frac{1}{y} + \frac{1}{x}.$$

So

$$\operatorname{curl} \mathbf{F} = \left(\frac{1}{x} + \frac{1}{y}\right) - \left(\frac{1}{y} + \frac{1}{x}\right) = 0$$

at every point with $x > 0$, $y > 0$. Since the region $x > 0$, $y > 0$ is simply-connected, the curl test guarantees existence of a potential function.

(d) We have

$$\frac{\partial F_2}{\partial x} = 2x\,e^{xy} + x^2 y\,e^{xy}, \qquad \frac{\partial F_1}{\partial y} = xe^{xy} + (1+xy)x\,e^{xy} = 2x\,e^{xy} + x^2 y e^{xy}.$$

The curl again vanishes at every point, and we conclude that the field is conservative.

3 (Curl Test II)**.** Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = \frac{1}{r}\mathbf{e}_\theta(r,\theta)$$

defined for (x,y) in $\mathbb{R}^2$ with $(x,y) \neq (0,0)$.

(a) Check that $\operatorname{curl}\mathbf{F} = 0$ for all $(x,y) \neq (0,0)$.

(b) Let C be the unit circle centered at the origin, oriented counterclockwise. Check that

$$\int_C \mathbf{F}\cdot\mathbf{dr} = 2\pi.$$

(c) The curl test seems to imply that $\mathbf{F}$ is conservative, as $\operatorname{curl}\mathbf{F} = 0$ at all points where $\mathbf{F}$ is defined by part (a). If $\mathbf{F}$ was conservative, we would have $\int_C \mathbf{F}\cdot\mathbf{dr} = 0$ for every closed curve C. Why doesn't part (b) then contradict the curl test?

Now, let $\mathbf{G}$ be the same vector field, but restricted to the smaller region $Y = \{(x,y) : x > 0\}$.

(d) Check that

$$\mathbf{G} = \nabla\left(\arctan\left(\frac{y}{x}\right)\right).$$

(e) Recall that $\arctan(y/x) = \theta(x,y)$ is the polar angle of the point (x,y). Conclude by the fundamental theorem of calculus for line integrals that for any curve C from point Q to point P in Y,

$$\int_C \mathbf{G}\cdot\mathbf{dr} = \theta(P) - \theta(Q).$$

Remark. For any closed curve, the integral

$$\frac{1}{2\pi}\int_C \mathbf{F}\cdot\mathbf{dr}$$

is called the *winding number* of C about the origin.

Solution.

(a) Computing,

$$\frac{\partial F_2}{\partial x} = \frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial F_1}{\partial y} = -\frac{1\cdot(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

So $\operatorname{curl}\mathbf{F} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0$ for all $(x,y) \neq (0,0)$.

(b) Parametrize the circle by $t \mapsto (\cos(t), \sin(t))$, $t \in [0, 2\pi]$. We have $\mathbf{v}(t) = (-\sin(t), \cos(t))$, and

$$\mathbf{F}(\mathbf{r}(t)) = \left(\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)} \right) = (-\sin(t), \cos(t)).$$

Taking the dot product,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = \sin^2(t) + \cos^2(t) = 1.$$

The work is equal to

$$\int_0^{2\pi} 1 \, dt = 2\pi.$$

(c) The vector field $\mathbf{F}$ is not defined at $(x, y) = (0, 0)$, and cannot be extended over that point (the magnitude goes to infinity). Since $\mathbb{R}^2$ without $(0, 0)$ is not simply connected, the curl test is not conclusive.

(d) Taking the partials,

$$\frac{\partial \arctan(y/x)}{\partial x} = \frac{1}{1 + (y/x)^2} \frac{-y}{x^2} = \frac{-y}{x^2 + y^2} = F_1,$$
$$\frac{\partial \arctan(y/x)}{\partial y} = \frac{1}{1 + (y/x)^2} \frac{1}{x} = \frac{x}{x^2 + y^2} = F_2.$$

Note $Y = \{(x, y) : x > 0\}$ does not contain $(0, 0)$, and is now simply-connected, so the curl test does apply, and should imply that the vector field is conservative — this is consistent with our finding.

(e) This follows immediately from the fundamental theorem for line integrals. The conclusion is interesting, however. The line integral $\int_C \mathbf{G} \cdot d\mathbf{r}$ measures the angle between the starting and ending points of the path C.

4 (Using Green's Theorem to Compute Area)**.** Define the following vector fields on $\mathbb{R}^2$:

$$\mathbf{F}_1(x, y) = \left(-\frac{y}{2}, \frac{x}{2} \right), \qquad \mathbf{F}_2(x, y) = (-y, 0), \qquad \mathbf{F}_3(x, y) = (0, x).$$

Let C be a simple closed curve, and let R be the region bounded by C. Orient C so that R appears on the left as one goes around C.

(a) Apply Green's Theorem to show that $\displaystyle\int_C \mathbf{F}_i \cdot d\mathbf{r} = \text{Area}(R)$ for each $i = 1, 2, 3$.

(b) (Ellipse) Find the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (try $\mathbf{F}_1$).

(c) (Arc of a Cycloid) Near the beginning of the course, we have seen that the path of a fixed point on the circumference of a unit circle rolling without slipping at unit speed may be parametrized by

$$t \mapsto (t - \sin(t),\, 1 - \cos(t)), \quad t \in \mathbb{R}.$$

As t varies in $[0, 2\pi]$, a single arc of the motion is traced out. Let C_1 denote this arc.

The curve C_1 is not closed. However, we can still apply Green's theorem to the piecewise curve $C = C_1 + C_2$, where C_2 is the line segment from $(2\pi, 0)$ to $(0,0)$! Compute $\int_C \mathbf{F_2} \cdot d\mathbf{r}$, and explain why this is equal to negative of the area under the arc of the cycloid.

(d) (The Folium of Descartes) Find the area of the region bounded by the loop of the folium of Descartes $x^3 + y^3 = 3xy$.

The loop may be parametrized (with orientation as in Green's theorem) by

$$t \mapsto \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right), \quad t \in [0, \infty)$$

(try $\mathbf{F}_3$ — the computation will take a little work).

Remark. The trick used in part (c) — closing up a curve to make it possible to apply Green's theorem — is a useful one.

Solution.

(a) We compute that

$$\begin{aligned}
\operatorname{curl} \mathbf{F}_1 &= \frac{\partial}{\partial x}\frac{x}{2} - \frac{\partial}{\partial y}\frac{-y}{2} = \frac{1}{2} - \frac{-1}{2} = 1, \\
\operatorname{curl} \mathbf{F}_2 &= \frac{\partial}{\partial x}0 - \frac{\partial}{\partial y} - y = 0 - (-1) = 1, \\
\operatorname{curl} \mathbf{F}_3 &= \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}0 = 1 - 0 = 1.
\end{aligned}$$

By Green's theorem for Work, for each $i = 1, 2, 3$,

$$\int_C \mathbf{F}_i \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}_i \, dA = \iint_R 1 \, dA = \operatorname{Area}(R).$$

(b) The ellipse may be parametrized by $t \mapsto (a\cos(t), b\sin(t))$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-a\sin(t), b\cos(t))$. We have

$$\mathbf{F}_1(\mathbf{r}(t)) \cdot \mathbf{v}(t) = \left(-\frac{b\sin(t)}{2}, \frac{a\cos(t)}{2} \right) \cdot (-a\sin(t), b\cos(t)) = \frac{ab}{2}.$$

Therefore,

$$\int_C \mathbf{F}_1 \cdot d\mathbf{r} = \int_0^{2\pi} \frac{ab}{2} \, dt = ab\pi.$$

(c) For C_1 (the cycloid arc), we have $\mathbf{v}(t) = (1 - \cos(t), \sin(t))$, and

$$\mathbf{F}_2(\mathbf{r}(t))\cdot\mathbf{v}(t) = (-(1-\cos(t), 0)\cdot(1-\cos(t), \sin(t) = -(1-\cos(t))^2 = -1+2\cos(t)-\cos^2(t).$$

The work is then

$$\int_{C_1} \mathbf{F_2} \cdot \mathbf{dr} = \int_0^{2\pi} -1 + 2\cos(t) - \cos^2(t)\, dt = -2\pi + 0 - \pi = -3\pi.$$

For C_2 (the segment along the x-axis), the work done is zero, since the field $\mathbf{F}_2$ vanishes along the x-axis. Indeed, parametrize as $t \mapsto (2\pi - t, 0)$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-1, 0)$, and

$$\mathbf{F}_2(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (0,0) \cdot (-1,0) = 0.$$

Therefore,

$$\int_C \mathbf{F}_2 \cdot \mathbf{dr} = \int_{C_1+C_2} \mathbf{F}_2 \cdot \mathbf{dr} = -3\pi + 0 = -3\pi.$$

Finally, since the curve C has the region it encloses on the right, to apply Green's theorem we need to reverse the orientation of C. We have seen that

$$\int_{-C} \mathbf{F}_2 \cdot \mathbf{dr} = -\int_C \mathbf{F}_2 \cdot \mathbf{dr} = 3\pi,$$

so the area enclosed by C is equal to 3π.

(d) The velocity of the provided parametrization is

$$\frac{dx}{dt} = \frac{3(1+t^3) - 3t(3t^2)}{(1+t^3)^2} = \frac{3-6t^3}{(1+t^3)^2},$$
$$\frac{dy}{dt} = \frac{6t(1+t^3) - 3t^2(3t^2)}{(1+t^3)^2} = \frac{6t-3t^4}{(1+t^3)^2}.$$

We have

$$\mathbf{F}_3(\mathbf{r}(t)) \cdot \mathbf{v}(t) = \left(0, \frac{3t}{1+t^3}\right) \cdot \mathbf{v}(t) = \frac{3t}{1+t^3}\frac{6t-3t^4}{(1+t^3)^2} = \frac{3t^2(6-3t^3)}{(1+t^3)^3}$$

and so the work integral is

$$\int_C \mathbf{F}_3 \cdot \mathbf{dr} = \int_0^\infty \frac{3t^2(6-3t^3)}{(1+t^3)^3}\, dt.$$

Let $u = 1 + t^3$, $du = 3t^2 dt$. Then the integral becomes

$$\int_1^\infty \frac{9-3u}{u^3}\, du = \int_1^\infty \frac{9}{u^3} - \frac{3}{u^2}\, du.$$

Both integrands decay rapidly as the upper limit is taken to infinity, so both converge and we can split the integral into two (it is also possible to find the antiderivative

$(-\frac{9}{2u^2} + \frac{3}{u})$ in a single step, so the splitting is not necessary, it is just to make the computations simpler to carry out and follow). For the first part,

$$\int_1^\infty \frac{9\,du}{u^3} = \lim_{M\to\infty} \int_1^M \frac{9\,du}{u^3} = \lim_{M\to\infty} \left[-\frac{9}{2}\frac{1}{u^2}\right]_1^M = \lim_{M\to\infty} \left(-\frac{9}{2}\frac{1}{M^2} + \frac{9}{2}\right) = \frac{9}{2}.$$

For the second part,

$$\int_1^\infty -\frac{3\,du}{u^2} = \lim_{M\to\infty} \int_1^M -\frac{3\,du}{u^2} = \lim_{M\to\infty} \left[\frac{3}{u}\right]_1^M = \lim_{M\to\infty} \left(\frac{3}{M} - 3\right) = -3.$$

So that

$$\text{area}(R) = \int_C \mathbf{F}_3 \cdot \mathbf{dr} = \frac{9}{2} - 3 = \frac{3}{2}.$$