MTHE 227 Problem Set 7 Solutions

1 (Jacobian of a Linear Map)**.** For $x(u,v) = au + bv$ and $y(u,v) = cu + dv$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus, the Jacobian of the map $T\colon \mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$ given by $(u,v) \mapsto (au + bv,\, cu + dv)$ is everywhere equal to T itself (and, as discussed in lecture, any linear map from $\mathbb{R}^2$ to itself can be written in this form). This fact is consistent with the intuition that the Jacobian of T at (u_0, v_0) is the linear map that best approximates T at (u_0, v_0). (If T is itself linear, then the best linear approximation is itself!)

Solution. This problem has an interesting conclusion, but the solution is quite short — we are simply asked to find four partial derivatives:

$$\begin{aligned}
\frac{\partial x}{\partial u} &= \frac{\partial(au+bv)}{\partial u} = a, \\
\frac{\partial x}{\partial v} &= \frac{\partial(au+bv)}{\partial v} = b, \\
\frac{\partial y}{\partial u} &= \frac{\partial(cu+dv)}{\partial u} = c, \\
\frac{\partial y}{\partial v} &= \frac{\partial(cu+dv)}{\partial v} = d.
\end{aligned}$$

2 (Geometry of Linear Maps)**.** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 matrix with $\det A = ad - bc \neq 0$. In linear algebra, one proves that A may be brought to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by finitely many of the following three operations (called elementary row operations):

(Op. 1) Switching two rows.

(Op. 2) Multiplying every entry of a row by a nonzero number.

(Op. 3) Adding a row to another row.

(More generally, any matrix may be brought to its reduced row-echelon form (rref) by a succession of the above three operations. All matrices with nonzero determinant have the identity matrix as their rref.)

(a) Define the following matrices:

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2(\lambda,1) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2(\lambda,2) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad E_3(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_3(2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Check that multiplying A on the left by:[1]

(i) E_1 switches the two rows of A;

(ii) $E_2(\lambda,1)$ multiplies every entry of the first row of A by λ;

(iii) $E_2(\lambda,2)$ multiplies every entry of the second row of A by λ;

(iv) $E_3(1)$ adds the second row to the first row; and

(v) $E_3(2)$ adds the first row to the second row.

(b) Conclude from part (a), and the linear algebra fact that A may be brought to the identity matrix by a finite sequence of elementary row operations, that there exists a sequence of matrices $M_1, \dots, M_r$, with each M_i being one of E_1, $E_2(\lambda,1)$, $E_2(\lambda,2)$, $E_3(1)$ or $E_3(2)$, so that

$$M_r\, M_{r-1} \cdots M_2\, M_1\, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Check that the inverses of the elementary matrices are:

$$E_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2(\lambda,1)^{-1} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2(\lambda,2)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \quad E_3(1)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_3(2)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Optional Problem. Check that the inverses of elementary matrices may be written in terms of elementary matrices:

$$\begin{aligned} E_1^{-1} &= E_1, \\ E_2(\lambda,1)^{-1} &= E_2\left(\frac{1}{\lambda}, 1\right), \\ E_2(\lambda,2)^{-1} &= E_2\left(\frac{1}{\lambda}, 2\right), \\ E_3(1)^{-1} &= E_2(-1,1)\, E_3(1)\, E_2(-1,1), \\ E_3(2)^{-1} &= E_2(-1,2)\, E_3(2)\, E_2(-1,2). \end{aligned}$$

(d) Recall that a 2×2 matrix defines a linear transformation on $\mathbb{R}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

For each of E_1^{-1}, $E_2(\lambda,1)^{-1}$, $E_2(\lambda,2)^{-1}$, $E_3(1)^{-1}$ and $E_3(2)^{-1}$, draw the image of the unit square $[0,1] \times [0,1]$ under the associated linear transformation. Identify each one as being a scaling, shear, or reflection about the diagonal $x = y$.

[1] Note: Multiplying A on the left by E_i means $E_i \cdot A$.

(e) Conclude that an arbitrary linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ with nonzero determinant may be realized as a composition of finitely many scalings, shears, and reflections about the diagonal.

(f) Draw the image of the unit square under each of the following linear maps, and decompose each of the linear maps into a sequence of scalings, shears, and reflections about the diagonal (row-reduce the matrix, keeping track of the steps!):

$$\text{(i)} \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution.

(a) For this part, we simply carry out the five matrix multiplications:

$$\text{(i)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0+c & 0+d \\ a+0 & b+0 \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

$$\text{(ii)} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a+0 & \lambda b+0 \\ 0+c & 0+d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix},$$

$$\text{(iii)} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ 0+\lambda c & 0+\lambda d \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix},$$

$$\text{(iv)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0+c & 0+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix},$$

$$\text{(v)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} a & b \\ a+b & c+d \end{pmatrix}.$$

(b) Since every operation of type (Op. 1), (Op. 2), (Op. 3) may be realized by a left-multiplication by one of the matrices E_1, $E_2(\lambda, 1)$, $E_2(\lambda, 2)$, $E_3(1)$ and $E_3(2)$, and every 2×2 matrix with nonzero determinant may be brought to the identity matrix by a finite number of operations of this type, we conclude that there exists a sequence of matrices as claimed.

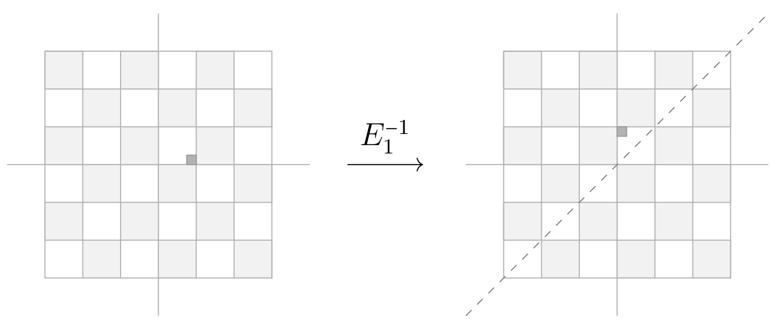
(c) More matrix multiplication! It is enough to check that $E_i^{-1} \cdot E_i = I$, where E_i is a type of elementary matrix, E_i^{-1} is its claimed inverse, and I is the 2×2 identity matrix (note: it is then automatically true that also $E_i \cdot E_i^{-1} = I$, a proposition from linear algebra of finite-dimensional vector spaces).

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0+1&0+0\\0+0&1+0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\begin{pmatrix}\frac{1}{\lambda}&0\\0&1\end{pmatrix}\begin{pmatrix}\lambda&0\\0&1\end{pmatrix}=\begin{pmatrix}\frac{\lambda}{\lambda}&0+0\\0+0&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0\\0&\frac{1}{\lambda}\end{pmatrix}\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}=\begin{pmatrix}1+0&0+0\\0+0&0+\frac{\lambda}{\lambda}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1+0&1-1\\0+0&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1+0&0+0\\-1+1&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

(d) The linear transformation $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ sends the unit vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

An arbitrary point (x, y) gets sent to (y, x). This is a reflection about the diagonal.

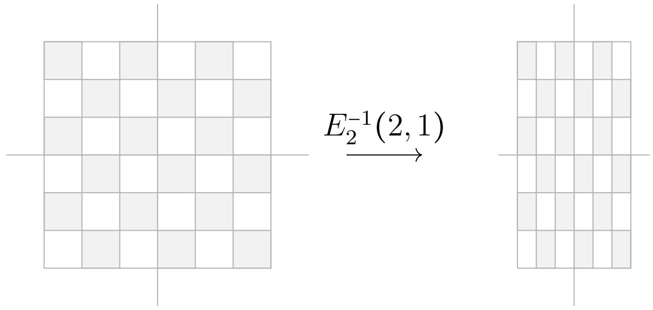


The linear transformation $\begin{pmatrix}\frac{1}{\lambda}&0\\0&1\end{pmatrix}$ sends the unit vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to

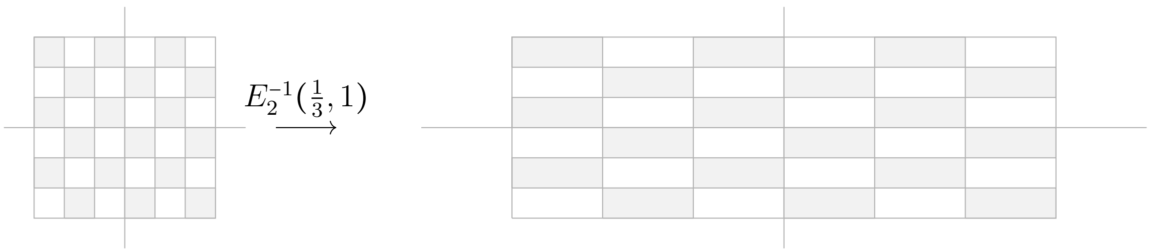
$$\begin{pmatrix}\frac{1}{\lambda}&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1/\lambda\\0\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}\frac{1}{\lambda}&0\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

This is a scaling by $1/\lambda$ along the x-coordinate.

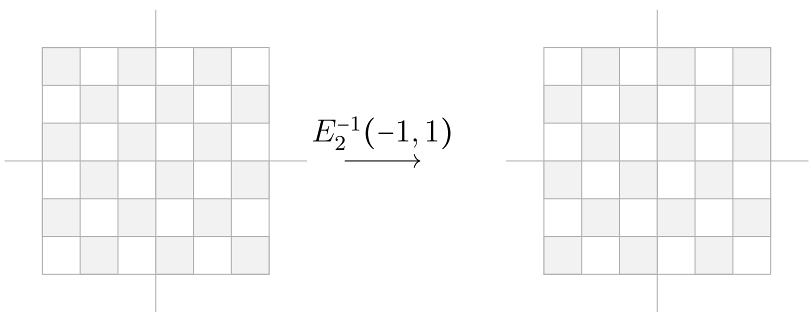
A few examples — for $\lambda = 2$, we have a shrinking along the x-coordinate by a factor of 2 (in other words, a scaling by a factor of 1/2):



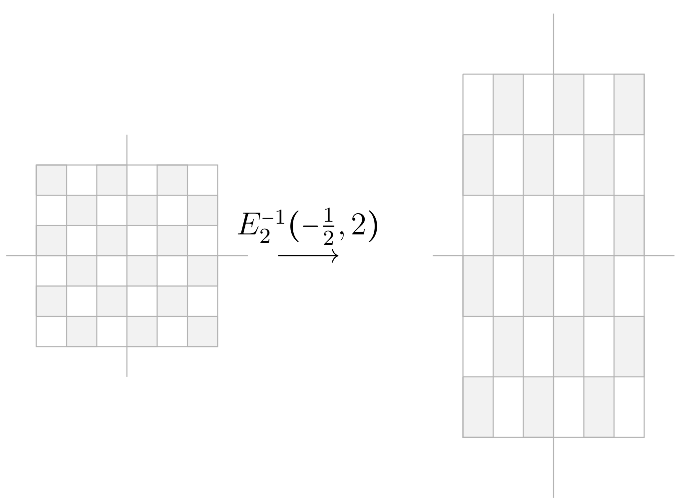
For $\lambda = 1/3$, we have an expansion along the x-coordinate by a factor of 3:



For the interesting case of $\lambda = -1$, we have a reflection about the y-axis (i.e. a scaling along the x-coordinate by a factor of -1):



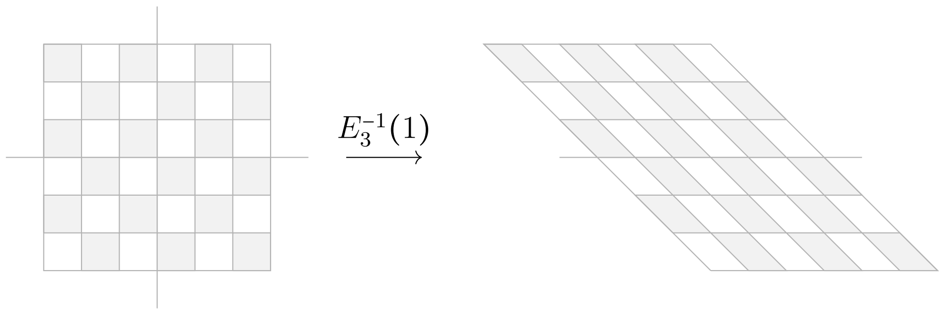
Similarly, $E_2^{-1}(\lambda, 2)$ is a scaling along the y-coordinate by $1/\lambda$. For example, for $\lambda = -\frac{1}{2}$, the transformation looks like a reflection about the x-axis and a stretch by a factor of 2:



The linear transformation $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ sends the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

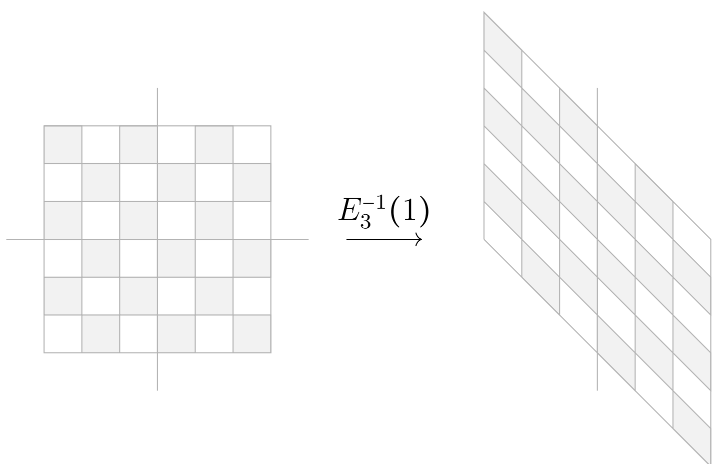
This is a shear parallel to the x-axis and going to the left:



Similarly, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ sends the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is a shear parallel to the y-axis and going down:



(e) From part (b), we have

$$M_r M_{r-1} \cdots M_2 M_1 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying on both sides by

$$(M_r M_{r-1} \cdots M_2 M_1)^{-1} = M_1^{-1} M_2^{-1} \cdots M_{r-1}^{-1} M_r^{-1},$$

we have

$$A = (M_r M_{r-1} \cdots M_2 M_1)^{-1} M_r M_{r-1} \cdots M_2 M_1 A = (M_r M_{r-1} \cdots M_2 M_1)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_1^{-1} M_2^{-1} \cdots M_{r-1}^{-1} M_r^{-1}.$$

From part (d), each M_i^{-1} is a scaling, shear, or reflection about the diagonal, and the claim follows.

(f) For both parts, we begin by row-reducing the matrix using (Op. 1), (Op. 2) and (Op. 3), keeping track of the steps. For part (i),

$$\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \xrightarrow{E_2(\frac{1}{2},1)} \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix} \xrightarrow{E_2(-1,2)} \begin{pmatrix} 1 & 0 \\ -1 & -6 \end{pmatrix} \xrightarrow{E_3(2)} \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix} \xrightarrow{E_2(-\frac{1}{6},2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$E_2(-\frac{1}{6}, 2)\, E_3(2)\, E_2(-1, 2)\, E_2(\frac{1}{2}, 1) \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

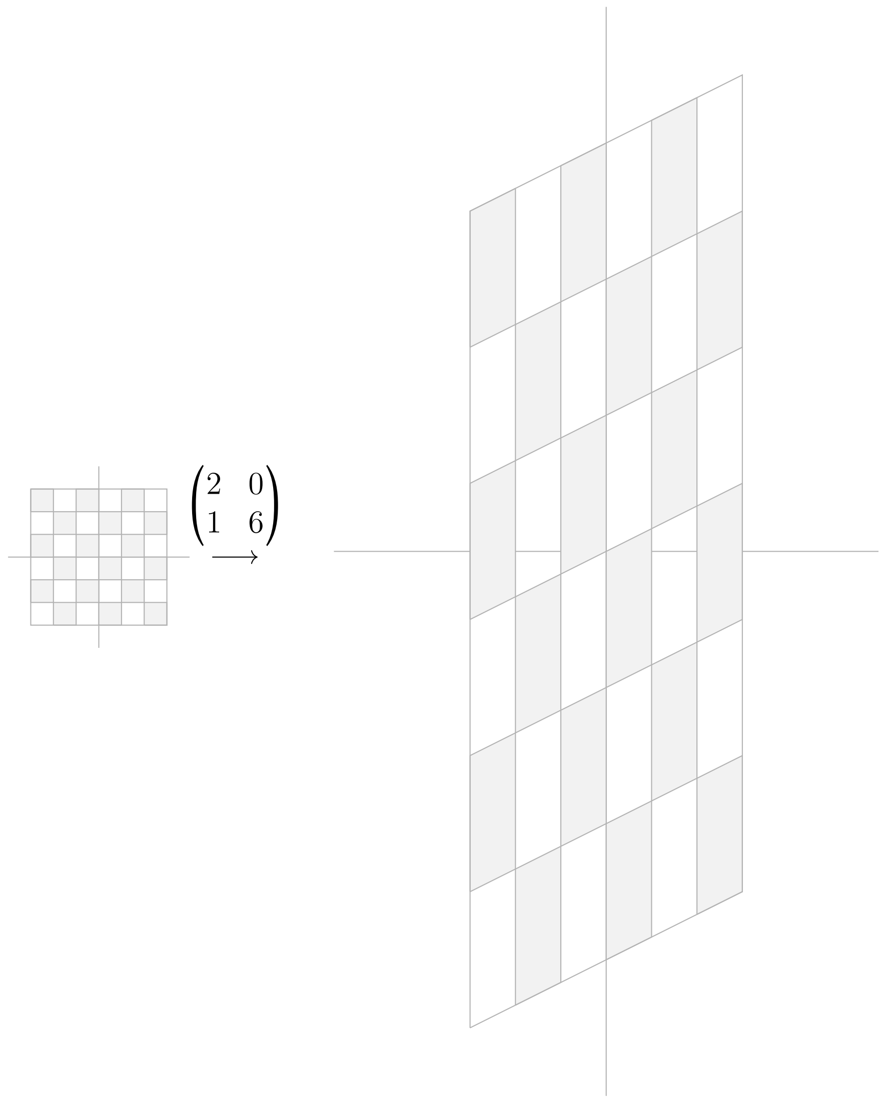
and so

$$\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} = E_2^{-1}(\frac{1}{2}, 1)\, E_2^{-1}(-1, 2)\, E_3^{-1}(2)\, E_2^{-1}(-\frac{1}{6}, 2).$$

The unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are sent to

$$\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

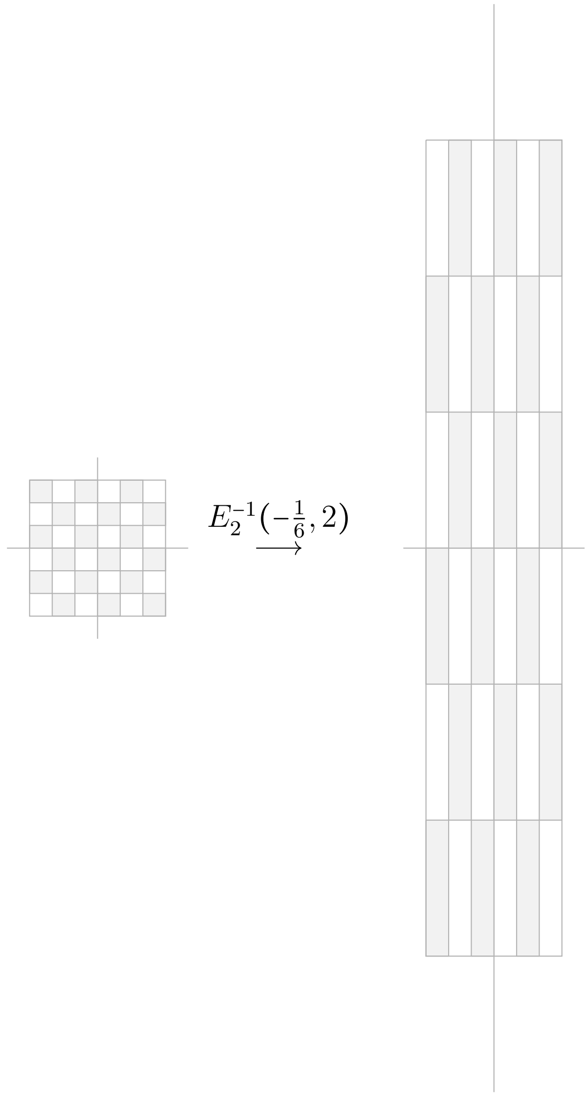
A sketch of the linear transformation is:



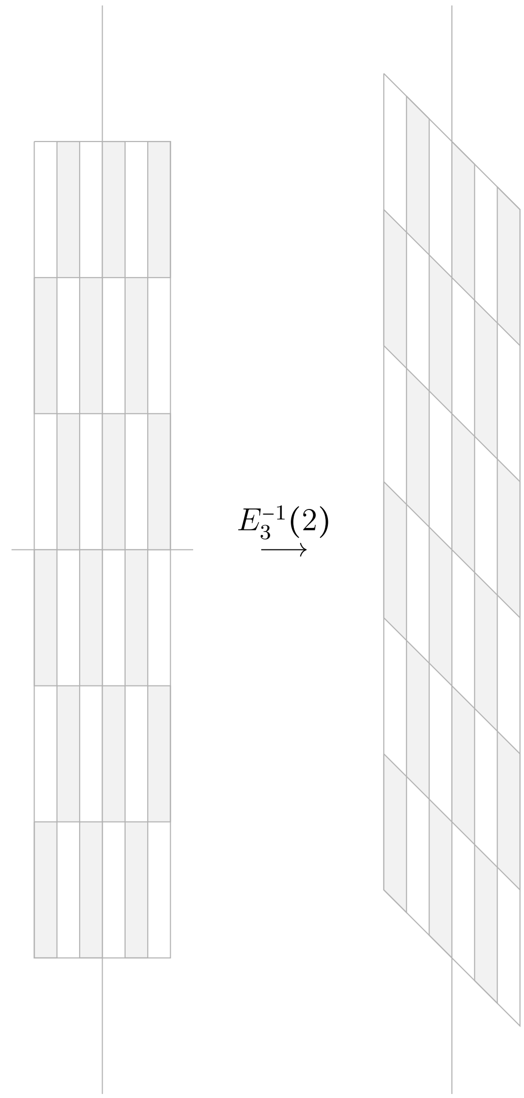
On the other hand, let's keep track what happens under the decomposition

$$\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} = E_2^{-1}(\frac{1}{2}, 1)\, E_2^{-1}(-1, 2)\, E_3^{-1}(2)\, E_2^{-1}(-\frac{1}{6}, 2).$$

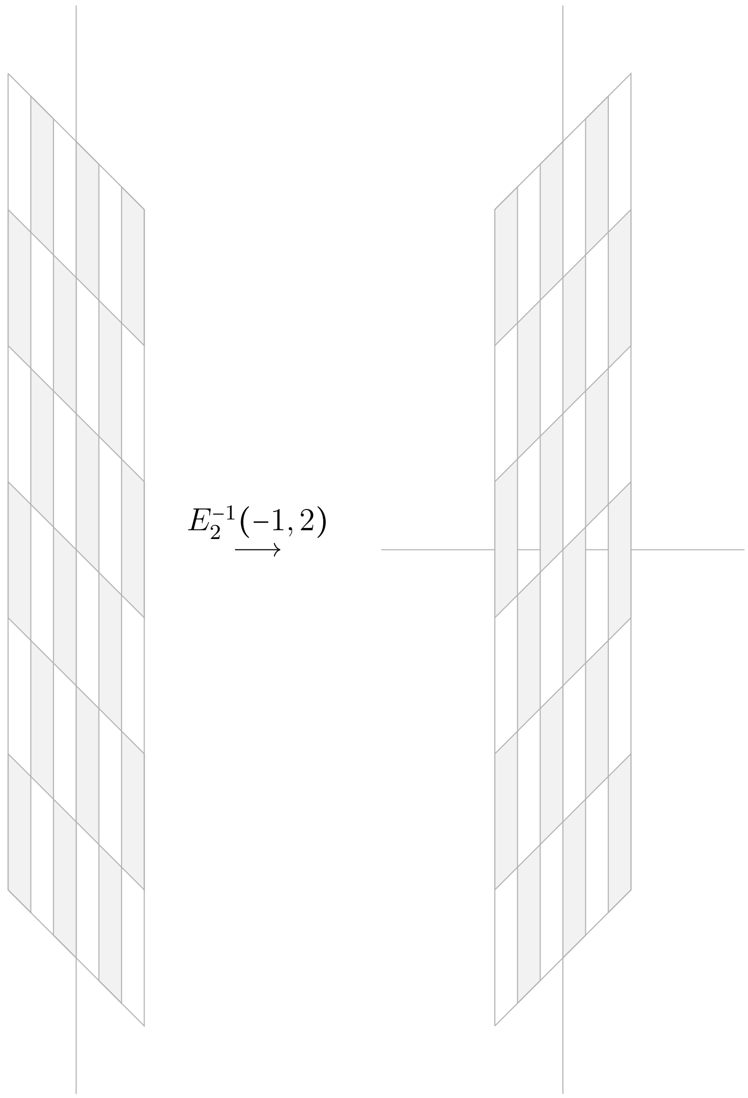
The linear transformation is applied right-to-left, so we begin with $E_2^{-1}(-\frac{1}{6}, 2)$, which is a scaling by -6 along the y-axis:



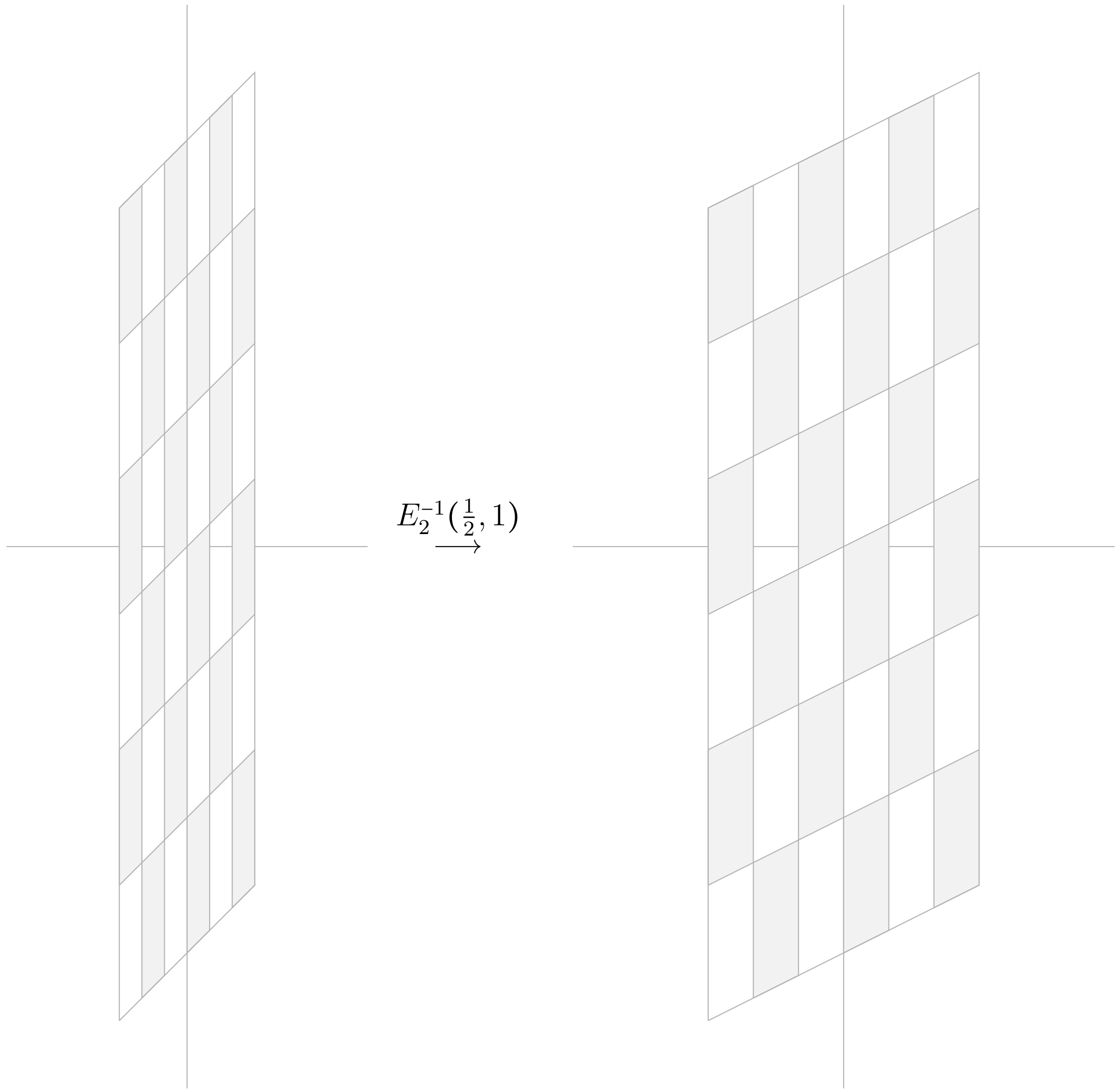
This is followed by $E_3^{-1}(2)$, which is a shear down:



Then, $E_2^{-1}(-1, 2)$ is a reflection about the x-axis (or scaling by -1 along y-coordinate):



Finally, $E_2^{-1}(\frac{1}{2}, 1)$ is a stretch by 2 along the x-axis:



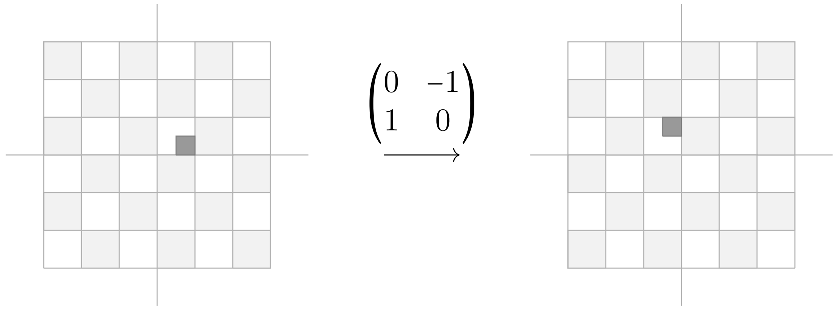
For part (ii), the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ row-reduces as:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{E_2(-1,2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we get the decomposition

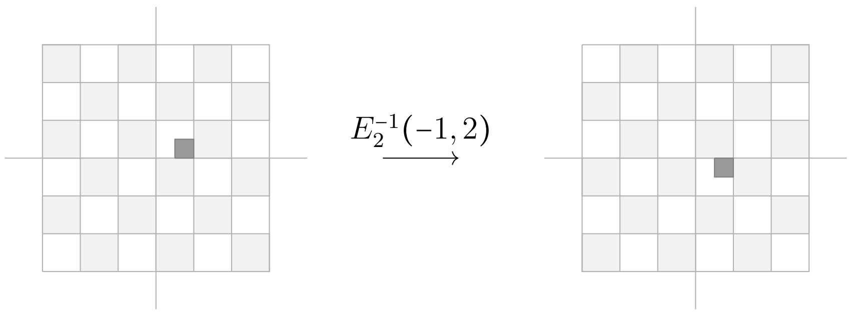
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = E_1^{-1} \, E_2^{-1}(-1, 2).$$

You may recognize from class that this is the matrix of a counterclockwise rotation by $\pi/2$ radians. The transformation looks like:

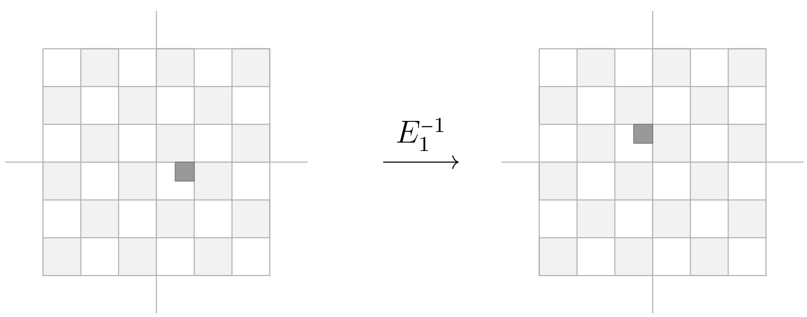


The decomposition into inverses of elementary matrices decomposes the rotation into two reflections.

First, $E_2^{-1}(-1,2)$ is a reflection about the x-axis:

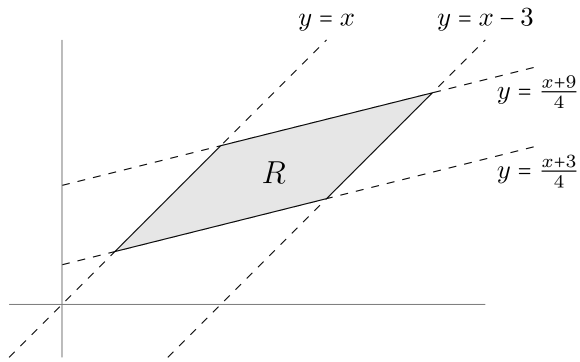


Then, E_1^{-1} is a reflection about the diagonal:



It is quite interesting that a rotation can be realized by two reflections.

3 (Double Integral Over a Parallelogram, Once Again)**.** Recall the parallelogram R with vertices $(1,1)$, $(3,3)$, $(5,2)$, $(7,4)$ from Problem Set 5:



The form of the equations of the boundary lines suggests that

$$u(x,y) = y - x, \qquad v(x,y) = y - \frac{x}{4}$$

is a good change of variables for this problem.

This describes the inverse map $T^{-1}\colon \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(u,v)}$.

(a) What is the region $R^* = T^{-1}(R)$ in $\mathbb{R}^2_{(u,v)}$?

(b) Solve for $x = x(u,v)$ and $y = y(u,v)$ as functions of u and v. (This amounts to finding the inverse of T^{-1}, or, in other words, finding T.)

(c) Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(d) Compute $\iint_R (y-x)^{2016}\, dA$ by applying the change of variables theorem.

Solution.

(a) The region R^* is the rectangle $[-3,\ 0] \times [3/4,\ 9/4]$.

There are many ways to justify this. For one, we have shown in lecture that linear maps take parallelograms to parallelograms, so the region R^* is a parallelogram, determined by its four vertices. The vertices of R go to

$$\begin{aligned}
(1,1) &\mapsto \left(1-1,\ 1-\frac{1}{4}\right) = \left(0,\ \frac{3}{4}\right), \\
(3,3) &\mapsto \left(3-3,\ 3-\frac{3}{4}\right) = \left(0,\ \frac{9}{4}\right), \\
(5,2) &\mapsto \left(2-5,\ 2-\frac{5}{4}\right) = \left(-3,\ \frac{3}{4}\right), \\
(1,1) &\mapsto \left(4-7,\ 4-\frac{7}{4}\right) = \left(-3,\ \frac{9}{4}\right).
\end{aligned}$$

These are the vertices of the rectangle $[-3,\ 0] \times [3/4,\ 9/4]$.

(b) A clean way to do this is to write down the matrix of T^{-1} and find its inverse.

The matrix of T^{-1} is

$$\begin{pmatrix} -1 & 1 \\ -\frac{1}{4} & 1 \end{pmatrix}.$$

The inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with nonzero determinant is $\frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore, the matrix of T is

$$\frac{1}{-1+\frac{1}{4}}\begin{pmatrix} 1 & -1 \\ \frac{1}{4} & -1 \end{pmatrix} = -\frac{4}{3}\begin{pmatrix} 1 & -1 \\ \frac{1}{4} & -1 \end{pmatrix}.$$

Writing out the variables, we have found that

$$x(u,v) = -\frac{4}{3}\left(u - v\right),$$
$$y(u,v) = -\frac{4}{3}\left(\frac{u}{4} - v\right).$$

For another possible solution, we could have played around with the equations. We have

$$v - u = (y - \frac{x}{4}) - (y - x) = \frac{3}{4}x,$$

so

$$x = -\frac{4}{3}(u - v);$$

and

$$4v - u = (4y - x) - (y - x) = 3y,$$

so

$$y = -\frac{4}{3}(\frac{u}{4} - v).$$

(For yet another possible approach, we could have converted the system of linear equations into matrix form and solved.)

(c) By Problem 1, the Jacobian of T is T itself.

$$\frac{\partial(x,y)}{\partial(u,v)} = T = -\frac{4}{3}\begin{pmatrix} 1 & -1 \\ \frac{1}{4} & -1 \end{pmatrix}.$$

The determinant of T (and therefore the determinant of the Jacobian) is equal to $-\frac{4}{3}$.

(d) By the change of variables theorem for double integrals (and Fubini's theorem),

$$\begin{aligned}
\iint_R (y-x)^{2016}\,dA &= \iint_{R^*} u^{2016} \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| \,dA \\
&= \int_{-3}^{0} \int_{3/4}^{9/4} u^{2016}\,\frac{4}{3}\,dv\,du \\
&= \frac{4}{3} \int_{-3}^{0} u^{2016} \left(\frac{9}{4} - \frac{3}{4}\right) du \\
&= \frac{4}{3}\,\frac{6}{4} \left[\frac{u^{2017}}{2017}\right]_{u=-3}^{u=0} \\
&= \frac{2 \cdot 3^{2017}}{2017}.
\end{aligned}$$

Remark. This example illustrates why it is necessary to take the absolute value of $\det \frac{\partial(x,y)}{\partial(u,v)}$ in the change of variables theorem — the transformation may be orientation-reversing (and so have a negative determinant), as is the case in this example!