MTHE 227 Problem Set 6 Solutions

Reminder. In lecture, we have defined the polar coordinate direction vector fields $\mathbf{e}_r$ and $\mathbf{e}_\theta$. These may be expressed in terms of the Cartesian direction vector fields (the latter also known as Cartesian direction vectors, the fields being constant) as [1]

$$\mathbf{e}_r(x,y) = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y = \frac{x\mathbf{e}_x + y\mathbf{e}_y}{\sqrt{x^2+y^2}},$$
$$\mathbf{e}_\theta(x,y) = -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y = \frac{-y\mathbf{e}_x + x\mathbf{e}_y}{\sqrt{x^2+y^2}}.$$

Going the other way, we have

$$\mathbf{e}_x(r,\theta) = \cos\theta\,\mathbf{e}_r(r,\theta) - \sin\theta\,\mathbf{e}_\theta(r,\theta),$$
$$\mathbf{e}_y(r,\theta) = \sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta).$$

Intuitively, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are steps of unit length in the directions of increasing r and θ, respectively.

1 (Velocity and Acceleration in Polar Coordinates)**.** We have seen that for a path parametrized in polar coordinates by $t \mapsto (r(t), \theta(t))$, t in $[0, 2\pi]$, the velocity and acceleration may be computed as

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r(t),\theta(t)) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(r(t),\,\theta(t)), \qquad \text{and}$$
$$\mathbf{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)).$$

To gain some understanding of the meaning of the various terms in the expression for the acceleration, for each of the following paths: sketch the path, compute the velocity and acceleration in polar coordinates, and sketch the velocity and acceleration vectors at a few points.

(a) (Accelerating Linear Motion) The path $t \mapsto (t^2, \pi/4)$, $t > 0$.

(b) (Uniform Circular Motion) The path $t \mapsto (R,\, 2016\,t)$, $t \in \mathbb{R}$. For this path, check that

$$\|\mathbf{a}(t)\| = \frac{\|\mathbf{v}(t)\|^2}{R} \quad \text{for all } t.$$

(This example is meant to shed some light on the $-r(d\theta/dt)^2$ term.)

(c) (Accelerating Circular Motion) The path $t \mapsto (R,\, 1008t^2)$, $t \in \mathbb{R}$.

[1] The second equalities hold as long as $(x,y) \neq (0,0)$.

(d) (Archimedean Spiral) The path $t \mapsto (t, t),\ t > 0$.

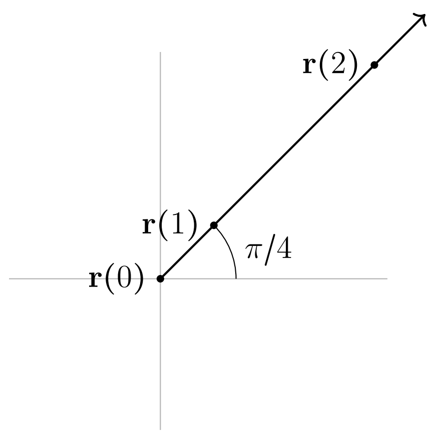
(One can think of this example as the path followed by a ball rolling radially at unit speed on a platform rotating with unit angular speed, from the reference frame of someone not standing on the platform. It is one of the simplest examples in which the $2\frac{dr}{dt}\frac{d\theta}{dt}$ term is nonzero.)

(e) (A Cardioid) The path $t \mapsto (1 + \cos(t),\ t) = (r(t), \theta(t)),\quad t \in [0, 2\pi]$. This is one possible parametrizations of the cardioid from Problem Set 5. Sketch the velocity and acceleration at $t = 0$, $t = \pi/4$, $t = \pi/2$ and $t = \pi$.

Solution.

(a) The path accelerates along a ray that makes an angle of $\pi/4$ with the x-axis. Positions at times $t = 0$, $t = 1$ and $t = 2$ are:

$$\mathbf{r}(0) = (0, \pi/4), \qquad \mathbf{r}(1) = (1, \pi/4), \qquad \mathbf{r}(2) = (4, \pi/4).$$

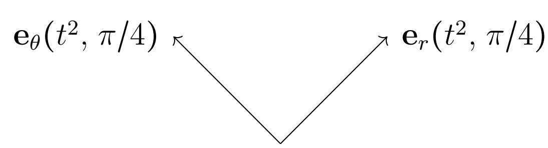


We compute

$$\frac{dr}{dt} = 2t, \quad \frac{d\theta}{dt} = 0,$$

$$\frac{d^2r}{dt^2} = 2, \quad \frac{d^2\theta}{dt^2} = 0.$$

At each point along the path, the direction vectors look like

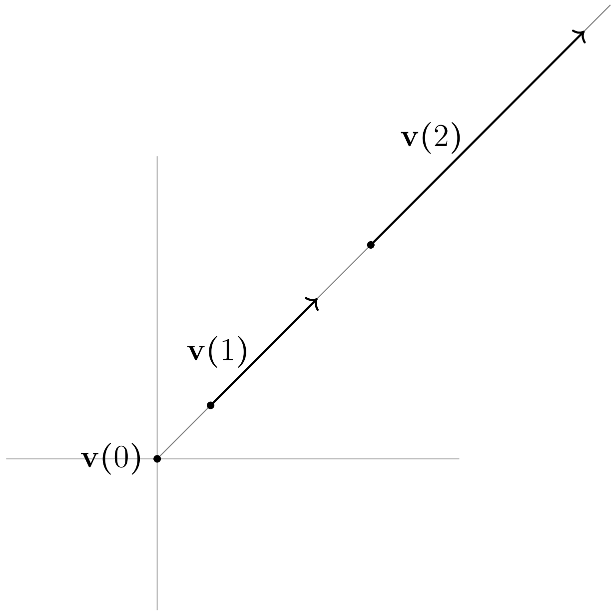


The velocity is

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(t^2, \pi/4) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(t^2, \pi/4) = 2t\,\mathbf{e}_r(t^2, \pi/4).$$

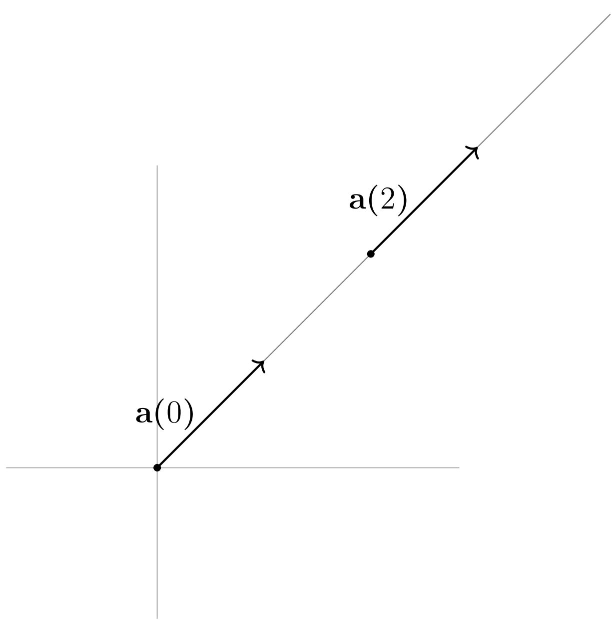
At times $t = 0$, $t = 1$, $t = 2$, the velocities are

$$\mathbf{v}(0) = 0, \qquad \mathbf{v}(1) = 2\,\mathbf{e}_r(1, \pi/4), \qquad \mathbf{v}(2) = 4\,\mathbf{e}_r(4, \pi/4).$$



The acceleraton is

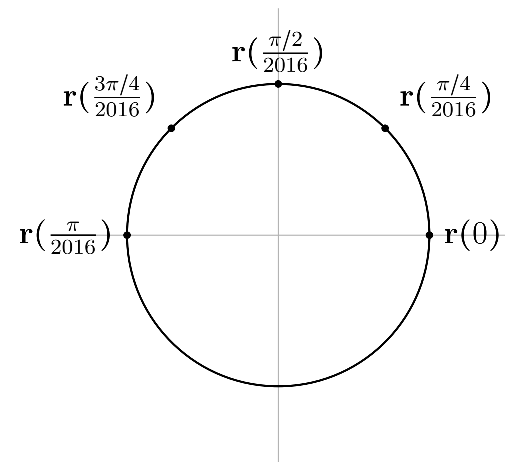
$$\mathbf{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(t^2, \pi/4) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(t^2, \pi/4) = 2\,\mathbf{e}_r(t^2,\ \pi/4).$$



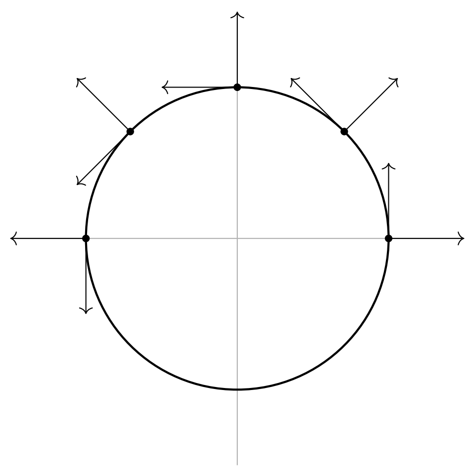
($\mathbf{a}(1)$ was skipped to avoid overlaps.)

(b) The path moves along a circle with constant speed. The position vectors at times $t = 0$, $t = \frac{\pi/4}{2016}$, $t = \frac{\pi/2}{2016}$, $t = \frac{3\pi/4}{2016}$ and $t = \frac{\pi}{2016}$ are

$$\mathbf{r}(0) = (R,\, 0), \qquad \mathbf{r}(\frac{\pi/4}{2016}) = (R,\, \pi/4), \qquad \mathbf{r}(\frac{\pi/2}{2016}) = (R,\, \pi/2),$$
$$\mathbf{r}(\frac{3\pi/4}{2016}) = (R,\, 3\pi/4), \qquad \mathbf{r}(\frac{\pi}{2016}) = (R,\, \pi).$$



The direction vectors at these positions look like



We compute

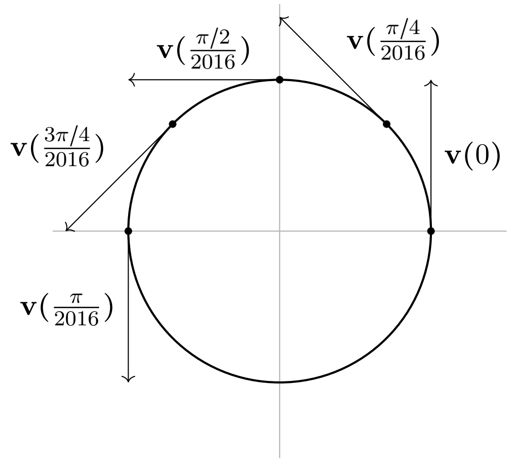
$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = 2016,$$

$$\frac{d^2r}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = 0.$$

The velocity is

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r(t),\theta(t)) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(r(t),\theta(t)) = 2016R\,\mathbf{e}_\theta(R, 2016t).$$

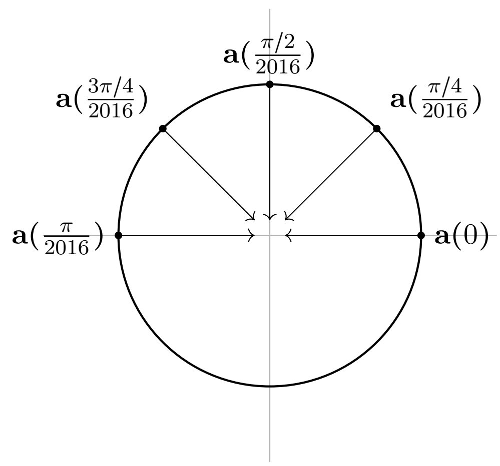
Not to scale (but with correct relative lengths), the velocities at the times above look like



The acceleraton is

$$\mathbf{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)) = -2016^2 R\,\mathbf{e}_r(R,\,2016t).$$

Again, the lengths of the following are not to scale (but all have equal lengths, so have the right relative scale):



For this path, we have

$$\begin{aligned}\|\mathbf{a}(t)\|^2 &= (-2016^2R\,\mathbf{e}_r(r(t),\theta(t)) + 0\,\mathbf{e}_\theta(r(t),\theta(t)))\cdot(-2016^2R\,\mathbf{e}_r(r(t),\theta(t)) + 0\,\mathbf{e}_\theta(r(t),\theta(t)))\\ &= 2016^4R^2,\end{aligned}$$

so that

$$\|\mathbf{a}(t)\| = 2016^2 R$$

(it is also possible to see this more geometrically — $\mathbf{a}(t)$ always points opposite to a direction vector, with magnitude 2016^2R.)

On the other hand, for this path

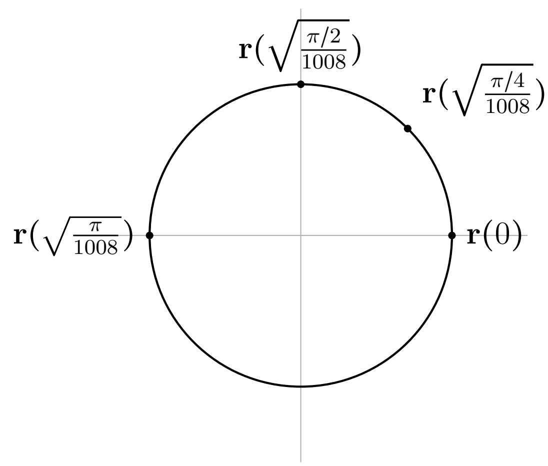
$$\begin{aligned}\|\mathbf{v}(t)\|^2 &= (0\,\mathbf{e}_r(r(t),\theta(t)) + 2016R\,\mathbf{e}_\theta(r(t),\theta(t)))\cdot(0\,\mathbf{e}_r(r(t),\theta(t)) + 2016R\,\mathbf{e}_\theta(r(t),\theta(t)))\\ &= 2016^2R^2\end{aligned}$$

So we see that

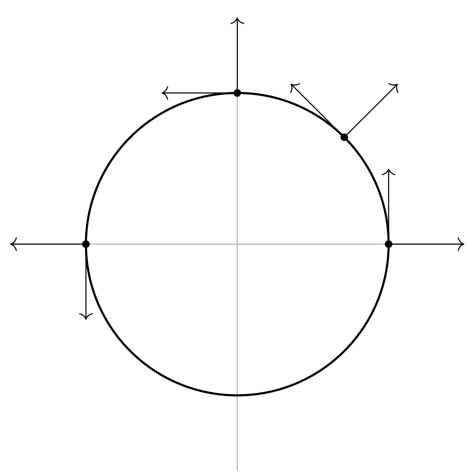
$$\|\mathbf{a}(t)\| = \frac{\|\mathbf{v}(t)\|^2}{R}.$$

(c) The path again goes along a circle of radius R, but this time with accelerating speed. At times $t = 0$, $\sqrt{\frac{\pi/4}{1008}}$, $\sqrt{\frac{\pi/2}{1008}}$, $\sqrt{\frac{\pi}{1008}}$, the positions are

$$\mathbf{r}(0) = (R, 0),$$
$$\mathbf{r}\left(\sqrt{\frac{\pi/4}{1008}}\right) = (R, \pi/4),$$
$$\mathbf{r}\left(\sqrt{\frac{\pi/2}{1008}}\right) = (R, \pi/2),$$
$$\mathbf{r}\left(\sqrt{\frac{\pi}{1008}}\right) = (R, \pi)$$



The direction vectors at these positions look like



We compute

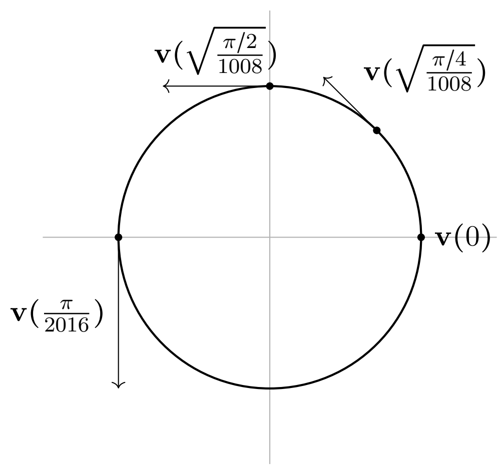
$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = 2016t,$$

$$\frac{d^2r}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = 2016.$$

The velocity is

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r(t),\theta(t)) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(r(t),\theta(t)) = 2016tR\,\mathbf{e}_\theta(R, 1008t^2).$$

Not to scale, the velocities at the times above look like

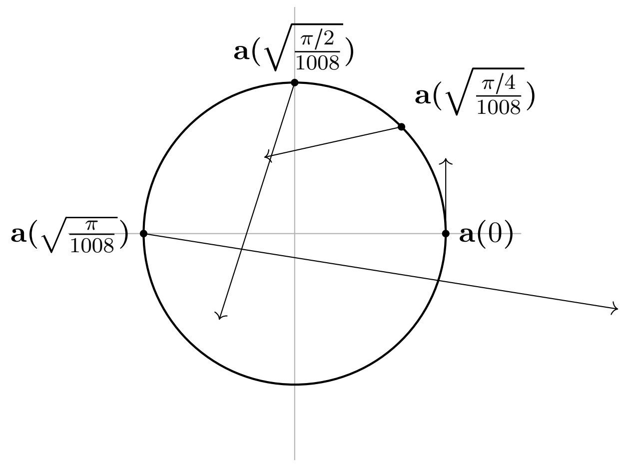


The vectors are increasing in length, but are always tangent to the circle.

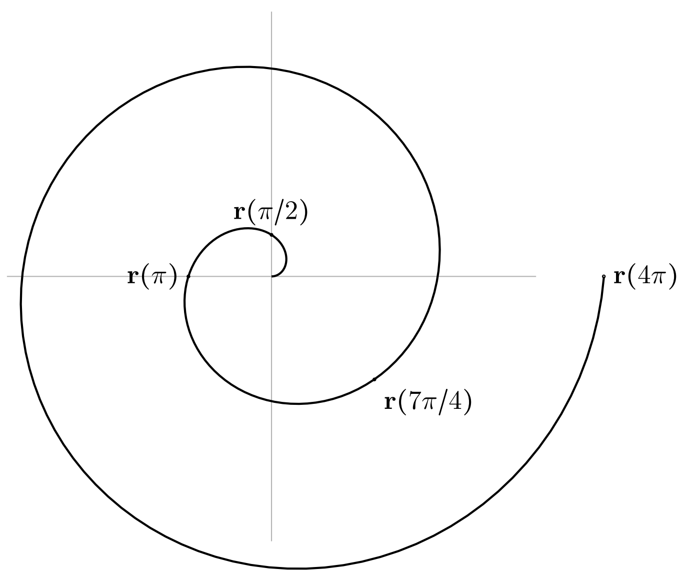
The acceleration is

$$\begin{aligned}\mathbf{a}(t) &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)) \\ &= -2016^2t^2R\;\mathbf{e}_r(R,\,1008t^2) + 2016R\;\mathbf{e}_\theta(R,\,1008t^2).\end{aligned}$$

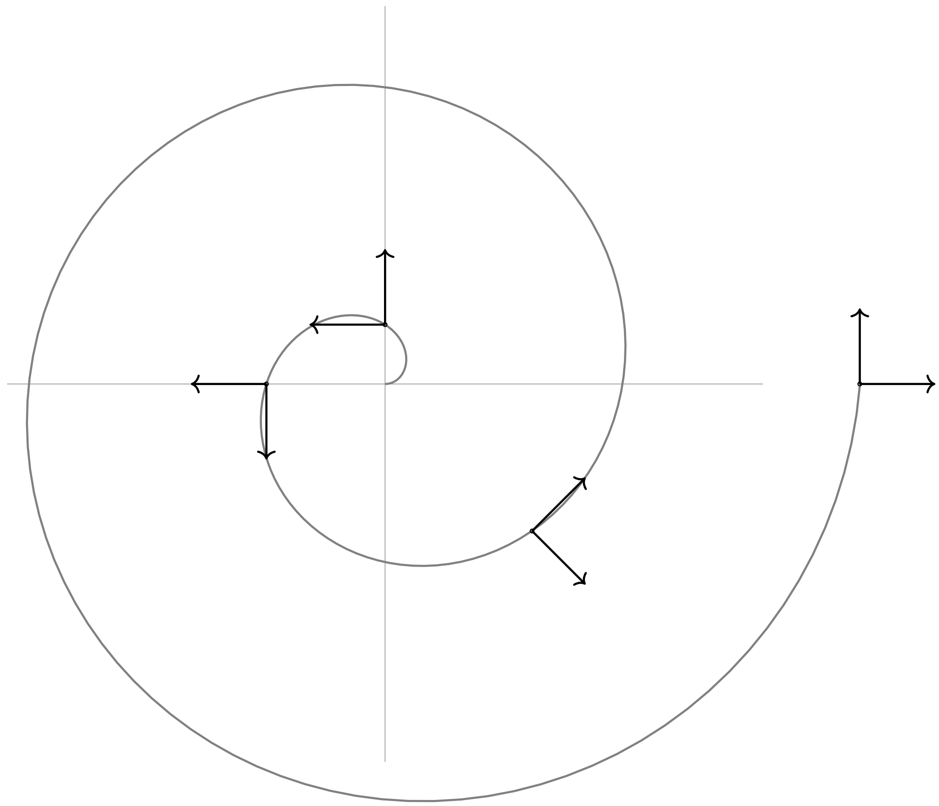
The acceleration now has a nonzero $\mathbf{e}_\theta$ term in its expansion, as well as dependence on t in the coefficient of $\mathbf{e}_r$.



(d) The path looks like a spiral. Let's look at the velocity and acceleration at times $t = \pi/2, t = \pi, t = 7\pi/4, t = 4\pi$.



The direction vector fields look as follows at these points (scaled up by a factor of 2 to make them easier to see):



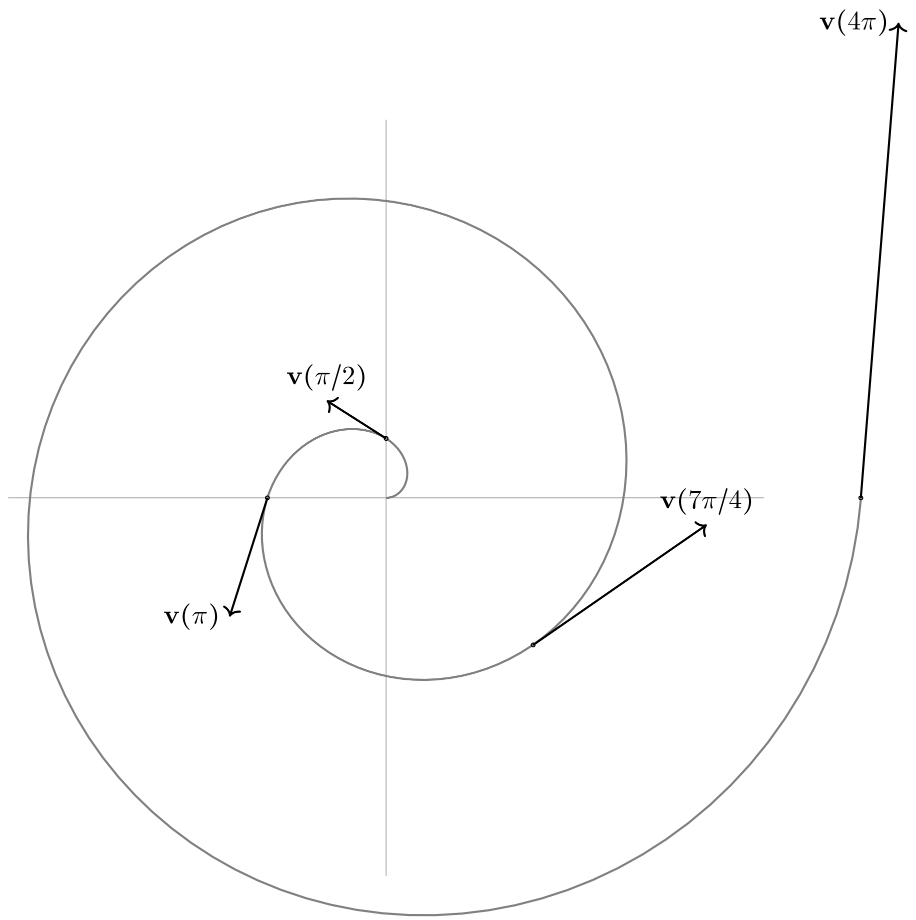
We compute

$$\frac{dr}{dt} = 1, \quad \frac{d\theta}{dt} = 1,$$

$$\frac{d^2r}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = 0.$$

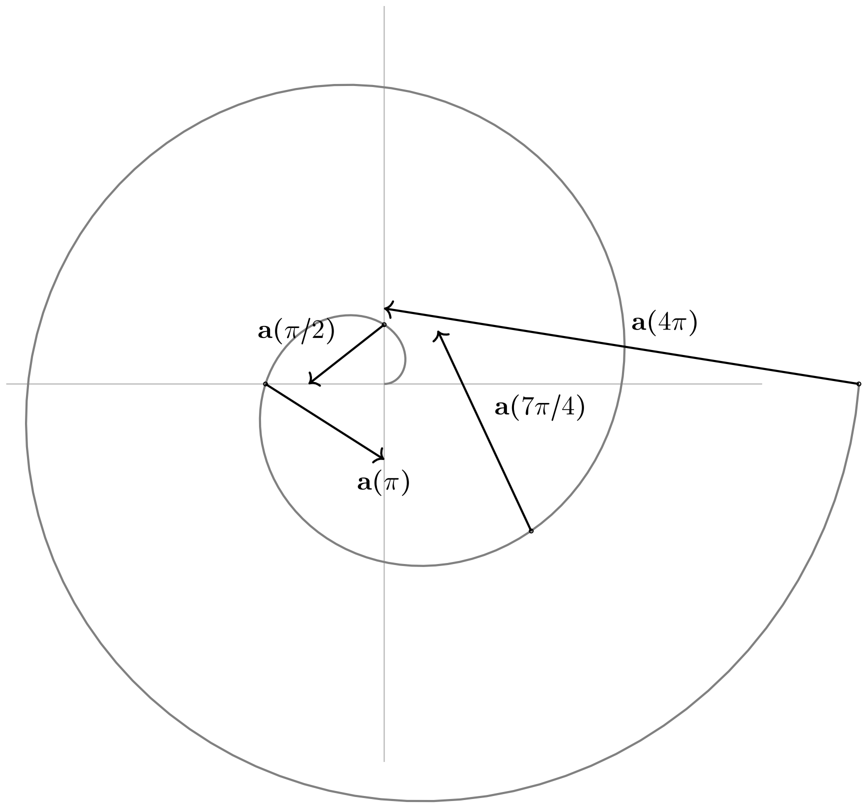
The velocity is

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(t,t) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(t,t) = \mathbf{e}_r(t,t) + t\mathbf{e}_\theta(t,t).$$

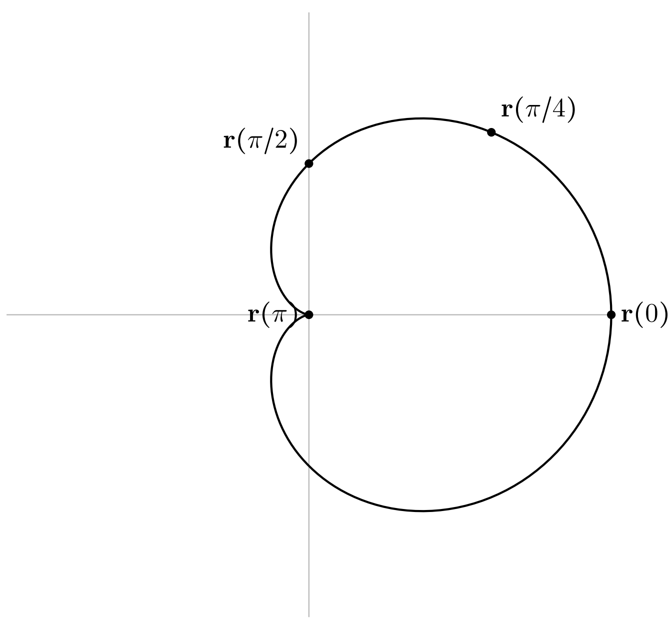


The acceleration is

$$\begin{aligned}\mathbf{a}(t) &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)) \\ &= -t\,\mathbf{e}_r(t,t) + 2\mathbf{e}_\theta(t,t).\end{aligned}$$



(e) This is the heart-shaped path from Problem Set 5.



We compute

$$\frac{dr}{dt} = -\sin(t), \quad \frac{d\theta}{dt} = 1,$$

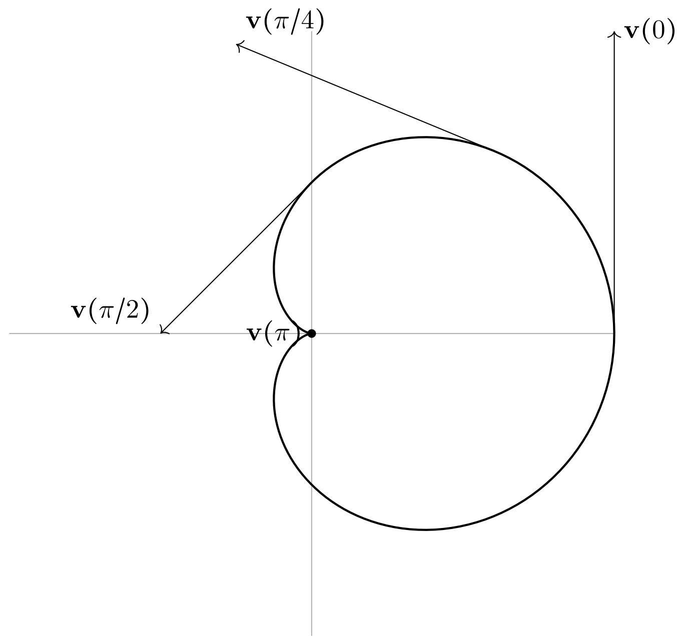
$$\frac{d^2r}{dt^2} = -\cos(t), \quad \frac{d^2\theta}{dt^2} = 0.$$

The velocity is

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(t,\,t) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(t,\,t) = -\sin(t)\,\mathbf{e}_r(1+\cos(t),\,t) + (1+\cos(t))\mathbf{e}_\theta(1+\cos(t),\,t).$$

At our points,

$$\begin{aligned}
\mathbf{v}(0) &= 2\,\mathbf{e}_\theta(\mathbf{r}(0)),\\
\mathbf{v}(\pi/4) &= -\frac{1}{\sqrt{2}}\,\mathbf{e}_r(\mathbf{r}(\pi/4)) + \frac{\sqrt{2}+1}{\sqrt{2}}\,\mathbf{e}_\theta(\mathbf{r}(\pi/4)),\\
\mathbf{v}(\pi/2) &= -1\,\mathbf{e}_r(\mathbf{r}(\pi/2)) + 1\,\mathbf{e}_\theta(\mathbf{r}(\pi/2)),\\
\mathbf{v}(\pi) &= 0.
\end{aligned}$$



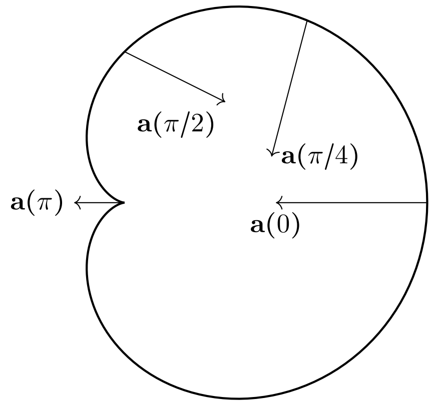
The acceleration is

$$\begin{aligned}
\mathbf{a}(t) &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t))\\
&= (-\cos(t) - (1+\cos(t)))\;\mathbf{e}_r(1+\cos(t),t) + -2\sin(t)\,\mathbf{e}_\theta(1+\cos(t),t)\\
&= -\,(1+2\cos(t))\;\mathbf{e}_r(1+\cos(t),t) + -2\sin(t)\,\mathbf{e}_\theta(1+\cos(t),t).
\end{aligned}$$

At our points,

$$\begin{aligned}
\mathbf{a}(0) &= -3\,\mathbf{e}_r(\mathbf{r}(0)),\\
\mathbf{a}(\pi/4) &= (-1-\sqrt{2})\,\mathbf{e}_r(\mathbf{r}(\pi/4)) - \sqrt{2}\,\mathbf{e}_\theta(\mathbf{r}(\pi/4)),\\
\mathbf{a}(\pi/2) &= -1\,\mathbf{e}_r(\mathbf{r}(\pi/2)) - 2\,\mathbf{e}_\theta(\mathbf{r}(\pi/2)),\\
\mathbf{a}(\pi) &= 1\,\mathbf{e}_r(\mathbf{r}(\pi)).
\end{aligned}$$

Scaled down by a factor of 3, the acceleration vectors look like:


$\mathbf{a}(\pi/2)$
$\mathbf{a}(\pi/4)$
$\mathbf{a}(\pi)$
$\mathbf{a}(0)$