MTHE 227 Problem Set 5 Solutions

1 (A Cardioid)**.** Let C be the closed curve in $\mathbb{R}^2$ whose polar coordinates (r,θ) satisfy

$$r = \cos\theta + 1.$$

(a) Sketch the curve C.

(b) Find a parametrization $t \mapsto (r(t), \theta(t))$, $t \in [a,b]$, of C in polar coordinates.

As we have seen in class, for a path parametrized in polar coordinates as in part (b), the arclength of the path can be computed by the expression

$$\int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt. \tag{1}$$

(c) Applying (1), or otherwise, show that the arclength of C is equal to 8. (The half-angle formula $|\cos(\frac{\theta}{2})| = \sqrt{\frac{1+\cos(\theta)}{2}}$ may come in useful. You can save yourself from dealing with the absolute value in the half-angle formula by noticing some symmetry in the problem.)

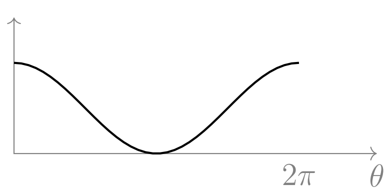
(d) Show that the area enclosed by C is equal to $3\pi/2$.

(e) Convert your parametrization of C found in (b) to Cartesian coordinates (x,y).

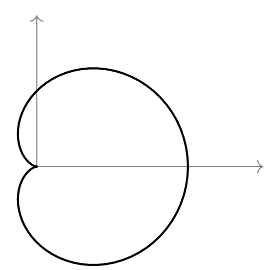
(C is a member of a family of curves cut out in polar coordinates by $r = 2a(\cos\theta + 1)$, with a a positive real number, called *cardioids*. The arclength of a general cardioid is $16a$, and its area is $6\pi a^2$. The name derives from the Greek word *kardia*, which translates to "heart".)

Solution.

(a) To draw the path, it is helpful to first draw the graph of $1 + \cos(\theta)$, on the left.



Graph of $1 + \cos(\theta)$



The Cardioid $r = 1 + \cos\theta$

(b) One possible parametrization is

$$t \mapsto (\cos(t) + 1,\, t) = (r(t),\, \theta(t)), \quad t \in [0, 2\pi].$$

For this parametrization, we have

$$\frac{dr}{dt} = -\sin(t),$$
$$\frac{d\theta}{dt} = 1.$$

(c) We have

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\sin^2(t)+(1+\cos(t))^2\cdot 1^2=\sin^2(t)+\cos^2(t)+2\cos(t)+1=2(1+\cos(t)),$$

so that

$$\sqrt{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2}=2\sqrt{\frac{1+\cos(t)}{2}}=2|\cos(\frac{t}{2})|.$$

The arclength integral is

$$\begin{aligned}\int_0^{2\pi}2|\cos(\frac{t}{2})|\,dt &= \int_0^{\pi}2\cos(\frac{t}{2})\,dt+\int_{\pi}^{2\pi}2(-\cos(\frac{t}{2}))\,dt\\ &= \left[4\sin(\frac{t}{2})\right]_0^{\pi}+\left[-4\sin(\frac{t}{2})\right]_{\pi}^{2\pi}\\ &= \left(4\sin(\frac{\pi}{2})-4\sin(0)\right)+\left(-4\sin(\pi)+4\sin(\frac{\pi}{2})\right)\\ &= 8.\end{aligned}$$

We could have also observed that by symmetry of $\cos(t)+1$, the arclength over the interval $0\le t\le\pi$ is equal to the arclength over $\pi\le t\le 2\pi$, so the arclength is equal to $2\int_0^{\pi}\cos(\frac{t}{2})\,dt=8$.

(d) Let R be the region enclosed by the cardioid. In polar coordinates,

$$R=\{(r,\theta):0\le r\le\cos(\theta)+1,\ 0\le\theta\le 2\pi\},$$

so that R is a Type II region in polar coordinates. By Fubini's theorem,

$$\begin{aligned}\iint_R 1\,dA &= \int_0^{2\pi}\int_0^{\cos(t)+1}1\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{cos(t)+1}d\theta\\ &= \int_0^{2\pi}\frac{\cos^2(t)+2\cos(t)+1}{2}\,d\theta\\ &= \int_0^{2\pi}\frac{\cos(2t)+4\cos(t)+3}{4}\,d\theta \qquad \text{using } \cos^2(t)=\frac{1+\cos(2t)}{2}\\ &= 0+0+\frac{3(2\pi-0)}{4}\\ &= \frac{3\pi}{2}.\end{aligned}$$

(e) The polar coordinates are related to Cartesian ones by $x=r\cos\theta$, $y=r\sin\theta$. Therefore, a possible parametrization of the cardioid in (x,y) coordinates is

$$x(t)=r(t)\cos(\theta(t))=(\cos(t)+1)\cos(t),\quad y(t)=r(t)\sin(\theta(t))=(\cos(t)+1)\sin(t),\quad t\in[0,2\pi].$$

2 (Averages of Averages)**.** Two common ways of finding the average of two nonnegative real numbers x and y are the *arithmetic mean*

$$\frac{x+y}{2}$$

and the *geometric mean*

$$\sqrt{xy}.$$

The two means are related by the (very useful) *arithmetic mean-geometric mean inequality*, or *AM-GM inequality* for short:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Fix a real number $m > 0$. If x and y are chosen independently and uniformly at random from the interval $[0, m]$ (meaning no value is more likely than any other), the expected values of the arithmetic and geometric means are given by

$$\iint_{[0,m]\times[0,m]} \frac{x+y}{2} \cdot \frac{1}{m^2}\, dA \quad \text{and} \quad \iint_{[0,m]\times[0,m]} \sqrt{xy} \cdot \frac{1}{m^2}\, dA,$$

respectively. Compute the two integrals. Are your findings consistent with the AM-GM inequality?

Solution. Both of the integrals have the square $[0, m] \times [0, m]$ as the region of integration, so by Fubini's theorem, we have

$$\begin{aligned}
\iint_{[0,m]\times[0,m]} \frac{x+y}{2} \frac{1}{m^2}\, dA &= \int_0^m \int_0^m \frac{x+y}{2} \frac{1}{m^2}\, dx dy \\
&= \frac{1}{m^2} \int_0^m \left[\frac{x^2}{4} + \frac{xy}{2}\right]_{x=0}^{x=m} dy \\
&= \frac{1}{m^2} \int_0^m \frac{m^2}{4} + \frac{my}{2}\, dy \\
&= \frac{1}{m^2} \left[\frac{m^2}{4} y + \frac{my^2}{4}\right]_{y=0}^{y=m} \\
&= \frac{1}{m^2} \left(\frac{m^3}{4} + \frac{m^3}{4}\right) \\
&= \frac{1}{m^2} \frac{m^3}{2} \\
&= \frac{m}{2}
\end{aligned}$$

and

$$\begin{aligned}
\iint_{[0,m]\times[0,m]} \sqrt{xy}\,\frac{1}{m^2}\,dA &= \int_0^m \int_0^m \sqrt{x}\sqrt{y}\,\frac{1}{m^2}\,dxdy \\
&= \frac{1}{m^2}\int_0^m \left[\frac{2}{3}x^{3/2}y^{1/2}\right]_{x=0}^{x=m} dy \\
&= \frac{1}{m^2}\int_0^m \frac{2}{3}m^{3/2}y^{1/2}\,dy \\
&= \frac{1}{m^2}\frac{2}{3}m^{3/2}\int_0^m y^{1/2}\,dy \\
&= \frac{1}{m^2}\frac{2}{3}m^{3/2}\left[\frac{2}{3}y^{3/2}\right]_{y=0}^{y=m} \\
&= \frac{1}{m^2}\frac{4}{9}m^3 \\
&= \frac{4m}{9}.
\end{aligned}$$

Because $\frac{x+y}{2} \geq \sqrt{xy}$ for any $x, y \geq 0$ (and multiplication by the positive constant $1/m^2$ preserves the inequality), we must have

$$\iint_{[0,m]\times[0,m]} \frac{x+y}{2}\,\frac{1}{m^2}\,dA \geq \iint_{[0,m]\times[0,m]} \sqrt{xy}\,\frac{1}{m^2}\,dA$$

Since $1/2 = 4/8 > 4/9$, the expected values of AM and GM computed above are indeed consistent with the AM-GM inequality.

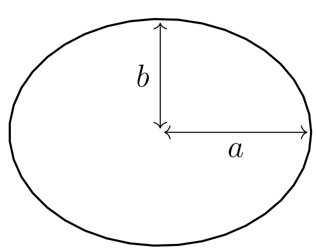
3 (Area of an Ellipse)**.** Find the area of the region R bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, by integrating the constant function 1 over R. What do you find in the case when $a = b$?

You may make use of the following without proof:

$$\int \sqrt{1-t^2}\,dt = \frac{t\sqrt{1-t^2} + \arcsin(t)}{2}.$$

Optional Problem. Integrate $\sqrt{1-t^2}$, by making the trigonometric substitution $t = \sin\theta$.

Solution. The ellipse is a Type III region, so we can integrate in either order. Let's arbitrarily choose x as the outer variable. From the standard form of the equation of the ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$), we can read off that the ellipse looks like



The Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Therefore, x runs from $(-a,0)$ to $(a,0)$ (you can quickly check that both of these points satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$). Then, solving the defining equation for y in terms of x, we have

$$y^2 = b^2(1-\frac{x^2}{a^2}),$$

so that the bounds on y are

$$-b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}.$$

The area integral is

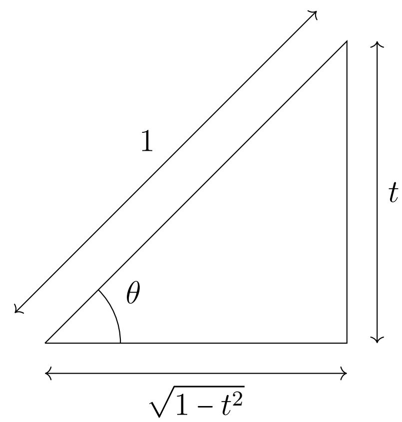
$$\begin{aligned}
\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dydx &= \int_{-a}^{a}\left(b\sqrt{1-\frac{x^2}{a^2}}+b\sqrt{1-\frac{x^2}{a^2}}\right)dx\\
&= 2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx. \quad \text{Letting } u = x/a,\ du = dx/a,\\
&= 2ab\int_{-1}^{1}\sqrt{1-u^2}\,du\\
&= 2ab\left[\frac{u\sqrt{1-u^2}+\arcsin(u)}{2}\right]_{u=-1}^{u=1}\\
&= 2ab\left(\frac{0+\pi/2}{2}-\frac{0-\pi/2}{2}\right) \quad \text{(the range of arcsin is } [-\pi/2,\pi/2]!)\\
&= 2ab\frac{\pi}{2}\\
&= ab\pi.
\end{aligned}$$

When $a=b$, we get the curve $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$, or $x^2+y^2=a^2$, which is a circle of radius a, and area $a\cdot a\cdot\pi = a^2\pi$, as expected.

Solution of the Optional Problem. Letting $t=\sin\theta$, $dt=\cos\theta\,d\theta$,

$$\begin{aligned}
\int\sqrt{1-t^2}\,dt &= \int\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta\\
&= \int\sqrt{\cos^2\theta}\,\cos\theta\,d\theta\\
&= \int\cos^2\theta\,d\theta\\
&= \int\frac{1+\cos(2\theta)}{2}\,d\theta\\
&= \frac{\theta}{2}+\frac{\sin(2\theta)}{4}\\
&= \frac{\theta+\sin(\theta)\cos(\theta)}{2} \quad \text{(using } \sin(2\theta)=2\sin(\theta)\cos(\theta)\text{).}
\end{aligned}$$

The following right triangle, constructed so that $\sin\theta = t/1 = t$, lets us find $\cos\theta$:



By the Pythagorean theorem, the length of the bottom side is equal to $\sqrt{1^2-t^2} = \sqrt{1-t^2}$, and $\cos\theta = \frac{\sqrt{1-t^2}}{1} = \sqrt{1-t^2}$. Therefore,

$$\int \sqrt{1-t^2}\,dt = \frac{\theta + \sin(\theta)\cos(\theta)}{2} = \frac{\arcsin(t) + t\sqrt{1-t^2}}{2}.$$

4. (a) For each of the following regions R, sketch R, and set up the double integral $\iint_R f(x,y)\,dA$ as an iterated integral (or possibly a sum of iterated integrals) in Cartesian coordinates. It is not necessary to evaluate the integral.

(i) The region R to the left of the y-axis and inside the circle $x^2+y^2=1$.

(ii) The region R bounded by the curves $x = y^2+3$ and $x = 4y^2$.

(iii) The region R bounded by the parallelogram with vertices $(1,1)$, $(3,3)$, $(5,2)$ and $(7,4)$.

(b) For each of the following, sketch the region of integration, switch the order of integration, and evaluate the integral.

(i) $\displaystyle\int_0^1 \left(\int_0^{\arcsin(x)} y^2\,dy\right) dx$

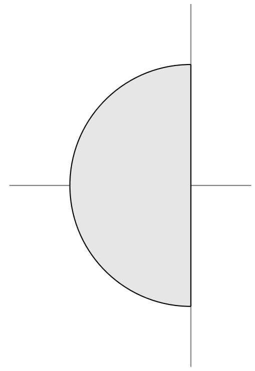
(ii) $\displaystyle\int_0^1 \left(\int_y^1 e^{x^2}\,dx\right) dy$ (Note: The antiderivative of e^{x^2} cannot be written down in terms of sums, products and powers of the usual functions x, $\cos(x)$, $\sin(x)$, $\exp(x)$, $\log(x)$, ... It has no antiderivative in elementary terms.)

(iii) $\displaystyle\int_{1/2}^1 \left(\int_1^{2y} \frac{\ln x}{x}\,dx\right) dy + \int_1^2 \left(\int_y^2 \frac{\ln x}{x}\,dx\right) dy.$

Solution.

(a) (i) The region is Type III. Integrating with respect to y first, the bounds on y are

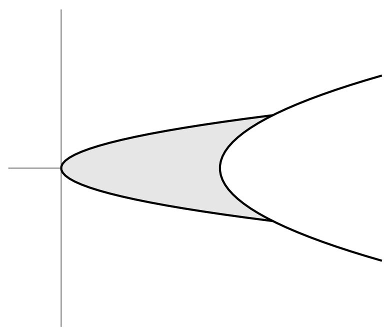
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$



By Fubini's theorem, the integral is

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dydx.$$

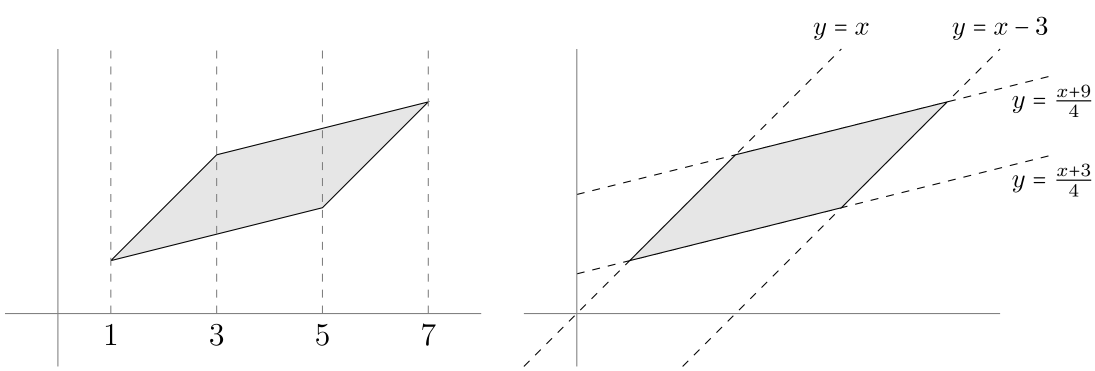
(ii) The two curves intersect when $4y^2 = y^2 + 3$, or $3y^2 = 3$, or $y = \pm 1$. The region bounded by the two curves looks like



This region is Type II, but not Type I. Therefore, we integrate with respect to x first. The integral is (applying Fubini's theorem, as usual)

$$\int_{-1}^{1}\int_{4y^2}^{y^2+3} f(x,y)\,dxdy.$$

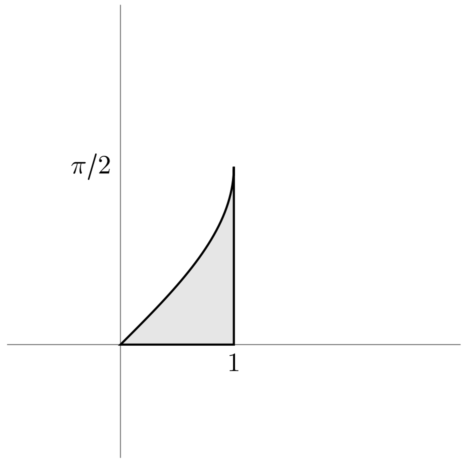
(iii) The region is Type III, but it is not easy to write the boundaries as a single explicit function. Therefore, we break it up into three parts, as indicated on the left diagram. The equations of the boundary lines are written down on the right diagram.



The integral is

$$\int_1^3 \int_{(x+3)/4}^{x} f(x,y)\,dydx + \int_3^5 \int_{(x+3)/4}^{(x+9)/4} f(x,y)\,dydx + \int_5^7 \int_{x-3}^{(x+9)/4} f(x,y)\,dydx.$$

(b) (i) The region of integration looks like



Therefore, the integral with switched order of integration is

$$\int_0^{\pi/2} \left(\int_{\sin(y)}^{1} y^2\,dx \right) dy = \int_0^{\pi/2} y^2 - y^2 \sin(y)\,dy.$$

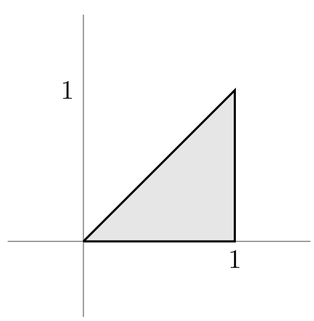
Integrating by parts twice, we have

$$\int y^2 \sin(y)\,dy = -y^2\cos(y) + \int 2y\cos(y)\,dy = -y^2\cos(y) + \left(2y\sin(y) - \int 2\sin(y)\,dy\right),$$

so that

$$\begin{aligned}\int_0^{\pi/2} y^2 - y^2\sin(y)\,dy &= \left[\frac{y^3}{3} + y^2\cos(y) - 2y\sin(y) - 2\cos(y)\right]_{y=0}^{y=\pi/2} \\ &= \left(\frac{\pi^3}{24} + 0 - \pi - 0\right) - (0 + 0 - 0 - 2) \\ &= \frac{\pi^3}{24} - \pi + 2.\end{aligned}$$

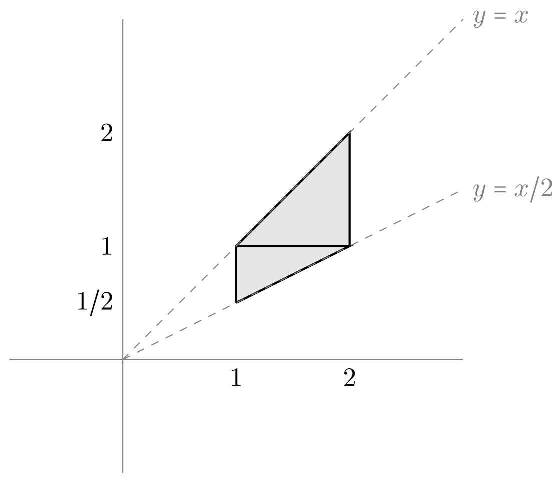
(ii) The region of integration looks like



Therefore, the integral with switched order of integration is

$$\int_0^1 \left(\int_0^x e^{x^2}\, dy \right) dx = \int_0^1 x e^{x^2}\, dx = \left[e^{x^2}/2 \right]_{x=0}^{x=1} = \frac{e-1}{2}.$$

(iii) The region of integration looks like



Therefore, the integral with switched order of integration is

$$\int_1^2 \left(\int_{x/2}^x \frac{\ln x}{x} dy \right) dx = \int_1^2 \ln x - \frac{\ln x}{2}\, dx = \frac{1}{2} \int_1^2 \ln x\, dx.$$

We have, by parts,

$$\int \ln x\, dx = x \ln(x) - \int\, dx = x \ln(x) - x,$$

so that

$$\int_1^2 \left(\int_{x/2}^x \frac{\ln x}{x} dy \right) dx = \frac{1}{2} \left[x \ln(x) - x \right]_{x=1}^{x=2} = \frac{1}{2} \left(2 \ln(2) - 2 + 1 \right) = \ln(2) - \frac{1}{2}.$$