MTHE 227 Problem Set 4 Solutions

1 (Finding Potentials)**.** For each of the following vector fields $\mathbf{F}$, find a real-valued function f such that $\mathbf{F} = \nabla f$ (problems from *Calculus* by J. Stewart):

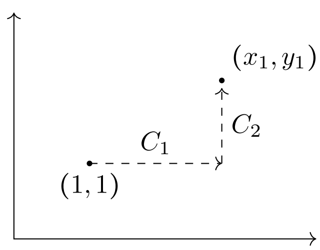
(a) $\mathbf{F}(x,y) = (\ \ln y + \frac{y}{x},\ \ln x + \frac{x}{y}\ )$ with $x > 0$, $y > 0$.

(b) $\mathbf{F}(x,y) = (\ (1+xy)e^{xy},\ x^2 e^{xy}\ )$ with $(x,y) \in \mathbb{R}^2$.

(c) $\mathbf{F}(x,y,z) = (\ y^2 z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2 z\ )$ with $(x,y,z) \in \mathbb{R}^3$.

Solution.

(a) Using 'Method 3' from lecture: take $(1,1)$ as the base point. Then, for any (x_1, y_1) in the region $\{(x,y) : x > 0,\ y > 0\}$, choose the path $C = C_1 + C_2$ from $(1,1)$ to (x_1, y_1), where C_1 is the straight line segment from $(1,1)$ to $(x_1, 1)$ and C_2 is the straight line segment from $(x_1, 1)$ to (x_1, y_1).



Then, define the potential at (x_1, y_1) by

$$f(x_1, y_1) = \int_C \mathbf{F} \cdot \mathbf{dr},$$

Parametrize C_1 and C_2 by

$$C_1 : t \mapsto (t, 1), \quad t \text{ from } 1 \text{ to } x_1,$$
$$C_2 : t \mapsto (x_1, t), \quad t \text{ from } 1 \text{ to } y_1.$$

For C_1:

The velocity vector is $\mathbf{v}(t) = (1,0)$ and so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (\ln 1 + \frac{1}{t},\ \ln t + \frac{t}{1}) \cdot (1,0) = \ln 1 + \frac{1}{t} = \frac{1}{t}.$$

We have

$$\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_1^{x_1} \frac{dt}{t} = [\ln t]_{t=1}^{t=x_1} = \ln x_1.$$

For C_2:

The velocity vector is $\mathbf{v}(t) = (0,1)$ and so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (\ln t + \frac{t}{x_1}, \ln x_1 + \frac{x_1}{t}) \cdot (0,1) = \ln x_1 + \frac{x_1}{t}.$$

We have

$$\int_{C_2} \mathbf{F} \cdot \mathbf{dr} = \int_1^{y_1} \ln x_1 + \frac{x_1}{t} \, dt = \left[t \ln x_1 + x_1 \ln t \right]_{t=1}^{t=y_1} = y_1 \ln x_1 + x_1 \ln y_1 - \ln x_1.$$

Finally,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_{C_1} \mathbf{F} \cdot \mathbf{dr} + \int_{C_2} \mathbf{F} \cdot \mathbf{dr} = \ln x_1 + y_1 \ln x_1 + x_1 \ln y_1 - \ln x_1 = y_1 \ln x_1 + x_1 \ln y_1$$

so that (we are finished integrating, and so drop the subscripts on x_1 and y_1)

$$f(x,y) = y \ln x + x \ln y$$

is a possible potential, as one can readily check.

We could have chosen different paths to (x_1, y_1).

Using 'Method 2': Taking the antiderivative of $\ln y + \frac{y}{x}$ with respect to x, we get $f(x,y) = x \ln y + y \ln x + g(y)$ is a potential, for some undetermined function $g(y)$. Taking the partial derivative of f with respect to y, we find that $\frac{x}{y} + \ln x + g'(y)$, so that $g'(y) = 0$, and the general potential is given by

$$x \ln y + y \ln x + C.$$

We could have also started by taking the antiderivative of $\ln x + \frac{x}{y}$ with respect to y. The computation would be similar.

(b) Using 'Method 3': take $(0,0)$ as the base point, and take the path $C = C_1 + C_2$ to (x_1, y_1) consisting of

$$C_1 : t \mapsto (t, 0), \quad t \text{ from } 0 \text{ to } x_1,$$
$$C_2 : t \mapsto (x_1, t), \quad t \text{ from } 0 \text{ to } y_1.$$

For C_1:

The velocity vector is $\mathbf{v}(t) = (1,0)$ and so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (\ (1 + t \cdot 0) e^{t \cdot 0}, t^2 e^{t \cdot 0}\ ) \cdot (1,0) = (1 + 0t) e^0 = 1.$$

We have

$$\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_0^{x_1} dt = x_1.$$

For C_2:

The velocity vector is $\mathbf{v}(t) = (0,1)$ and so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (\ (1 + x_1 t) e^{x_1 t}, x_1^2 e^{x_1 t}\ ) \cdot (0,1) = x_1^2 e^{x_1 t}.$$

We have

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{y_1} x_1^2 e^{x_1 t}\, dt = x_1 e^{x_1 y_1} - x_1.$$

So that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = x_1 + (x_1 e^{x_1 y_1} - x_1) = x_1 e^{x_1 y_1}.$$

One possible potential is

$$f(x,y) = xe^{xy}.$$

Using 'Method 2': If we begin by taking the indefinite integral with respect to y (which seems simpler), we get

$$f(x,y) = xe^{xy} + g(x),$$

where g is an undetermined function of x. Then, taking the partial with respect to x, we need $\partial f/\partial x = e^{xy} + xye^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. Comparing with the first component of the vector field $\mathbf{F}$, we see that $g'(x) = 0$, so that $g(x) = C$ and an arbitrary potential is given by

$$f(x,y) = xe^{xy} + C.$$

If we tried to take the indefinite integral with respect to x in the beginning, we would have ended up with a term of the form xye^{xy}, that needs to be integrated by parts.

(c) Using 'Method 3': take $(0,0,0)$ as the reference point, and take C to be the line segment connecting $(0,0,0)$ and (x_1,y_1,z_1), parametrized by $t \mapsto (\, x_1 t,\, y_1 t,\, z_1 t\,)$, $t \in [0,1]$. The velocity vector of the parametrization is $\mathbf{v}(t) = (x_1,\, y_1,\, z_1)$, independent of t. Therefore,

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t) &= \left(\, y_1^2 z_1 t^3 + 2x_1 z_1^2 t^3,\ 2x_1 y_1 z_1 t^3,\ x_1 y_1^2 t^3 + 2x_1^2 z_1 t^3 \,\right)\cdot(x_1,\, y_1,\, z_1)\\
&= \left(\, x_1 y_1^2 z_1 + 2x_1^2 z_1^2 + 2x_1 y_1^2 z_1 + x_1 y_1^2 z_1 + 2x_1^2 z_1^2 \,\right) t^3\\
&= \left(\, 4x_1^2 z_1^2 + 4x_1 y_1^2 z_1 \,\right) t^3\\
&= \left(\, x_1^2 z_1^2 + x_1 y_1^2 z_1 \,\right) 4t^3.
\end{aligned}$$

Computing the work, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(\, x_1^2 z_1^2 + x_1 y_1^2 z_1 \,\right) 4t^3\, dt = \left(\, x_1^2 z_1^2 + x_1 y_1^2 z_1 \,\right)\left[t^4\right]_{t=0}^{t=1}.$$

So that one possible potential is

$$f(x,y,z) = x^2 z^2 + xy^2 z.$$

This was comparatively short! We have taken advantage of the fact that the field is given by polynomials that are all degree 3.

Of course, we could have taken the usual $C = C_1 + C_2 + C_3$ path with parts that are parallel to the x-, y- and z-axes (as seen in class), which would have taken more work.

Using 'Method 2': Since the y-component of the vector field has only one term, let's begin by taking its antiderivative. We have

$$\frac{\partial f}{\partial y} = 2xyz \implies f(x,y,z) = xy^2z + g(x,z).$$

Taking the partial derivative of the result with respect to x, we find

$$\frac{\partial f}{\partial x} = y^2z + \frac{\partial g}{\partial x}$$

which should equal to $F_1(x,y,z) = y^2z + 2xz^2$. We conclude that

$$\frac{\partial g}{\partial x} = 2xz^2 \implies g(x,z) = x^2z^2 + h(z).$$

Plugging this back into the expression for f, we find

$$f(x,y,z) = xy^2z + x^2z^2 + h(z).$$

Taking the partial with respect to z,

$$\frac{\partial f}{\partial z} = xy^2 + 2x^2z + h'(z).$$

This should be equal to $F_3(x,y,z) = xy^2 + 2x^2z$, which implies that $h'(z) = 0$. In conclusion, a general potential is given by

$$f(x,y,z) = xy^2z + x^2z^2 + C.$$

2 (Geometric Meaning of Flux)**.** Let C be the unit circle in $\mathbb{R}^2$ with normal pointing away from the origin (note: unlike Problem Set 3, C is the entire circle, not just the upper semicircle). Define the following vector fields:

$$\mathbf{F}(x,y) := (x,y), \qquad \mathbf{G}(x,y) := \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right), \qquad \mathbf{H}(x,y) := (-y,x) \qquad \text{for all } (x,y) \in \mathbb{R}^2$$

(As a reminder, these three vector fields can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ in place counterclockwise by $0, \pi/4$ and $\pi/2$ radians, respectively.)

Compute the flux $\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$, $\int_C \mathbf{G}\cdot\hat{\mathbf{n}}\,ds$, and $\int_C \mathbf{H}\cdot\hat{\mathbf{n}}\,ds$ of each of the three vector fields across C. Which of the three is largest? Which is smallest? Explain briefly. (Be careful to orient the normals correctly.)

Optional Problem. The three vector fields above are members of the family

$$\mathbf{F}_\theta(x,y) = (x\cos\theta - y\sin\theta,\, x\sin\theta + y\cos\theta),$$

with $0 \le \theta < 2\pi$ ($\mathbf{F} = \mathbf{F}_0$, $\mathbf{G} = \mathbf{F}_{\pi/4}$, $\mathbf{H} = \mathbf{F}_{\pi/2}$). The vector field $\mathbf{F}_\theta$ can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ counterclockwise by θ radians (in place). Plot the flux of $\mathbf{F}_\theta$ across C, as a function of θ.

Solution. Parametrize the circle C by $t \mapsto (\cos t, \sin t)$, $t \in [0, 2\pi]$. The velocity of this parametrization is $\mathbf{v}(t) = (-\sin t, \cos t)$, hence the outward normal is $\mathbf{n}_+(t) = (y'(t), -x'(t)) = (\cos t, \sin t)$ (because the parametrization of C is going counterclockwise, we need to take $\mathbf{n}_+$ to get an outward normal). Notice that $\mathbf{n}_+(t) = \mathbf{r}(t)$ for each t, as we might have seen from geometry.

We have

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) &= (\cos t, \sin t) \cdot (\cos t, \sin t) = \cos^2 t + \sin^2 t = 1, \\
\mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) &= \left(\frac{\cos t - \sin t}{\sqrt{2}}, \frac{\cos t + \sin t}{\sqrt{2}} \right) \cdot (\cos t, \sin t) \\
&= \frac{1}{\sqrt{2}} \left(\cos^2 t - \sin t \cos t + \cos t \sin t + \sin^2 t \right) = \frac{1}{\sqrt{2}}, \\
\mathbf{H}(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) &= (-\sin t, \cos t) \cdot (\cos t, \sin t) = -\sin t \cos t + \cos t \sin t = 0.
\end{aligned}$$

Therefore,

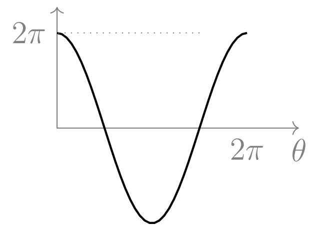
$$\begin{aligned}
\int_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds &= \int_0^{2\pi} dt = 2\pi, \\
\int_C \mathbf{G} \cdot \hat{\mathbf{n}} \, ds &= \int_0^{2\pi} \frac{1}{\sqrt{2}} \, dt = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi, \\
\int_C \mathbf{H} \cdot \hat{\mathbf{n}} \, ds &= \int_0^{2\pi} 0 \, dt = 0,
\end{aligned}$$

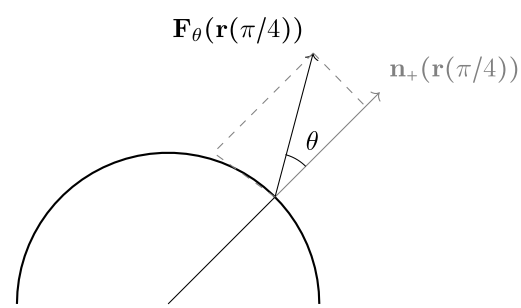
Geometrically, each of the three vector fields makes a constant angle with the outward normal (they are 0, $\pi/4$ and $\pi/2$ radians for $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ respectively). Since each vector of the three fields has length one on the unit circle C, it follows that the flux of the fields across C can be computed by multiplying the 'local flow' 1, $1/\sqrt{2}$, 0 of $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ by the arclength 2π of the unit circle C.

Solution of the Optional Problem. Using the same parametrization, we find

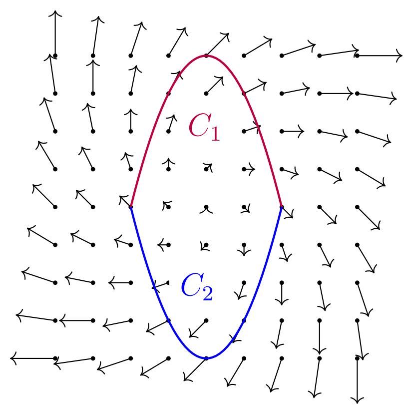
$$\begin{aligned}
\mathbf{F}_\theta(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) &= (\cos t \cos\theta - \sin t \sin\theta, \cos t \sin\theta + \sin t \cos\theta) \cdot (\cos t, \sin t) \\
&= \cos^2 t \cos\theta - \cos t \sin t \sin\theta + \cos t \sin t \sin\theta + \sin^2 t \cos\theta \\
&= (\cos^2 t + \sin^2 t) \cos\theta = \cos\theta.
\end{aligned}$$

Therefore, the flux $\int_C \mathbf{F}_\theta \cdot \hat{\mathbf{n}} \, ds$ is equal to $2\pi \cos\theta$. The geometric explanation is similar to that in the previous paragraph. Notice that the flux becomes negative when $\pi/2 < \theta < 3\pi/2$, when the flow is going against the normal.





3 (More Practice with Flux)**.** Let $C = C_1 + C_2$, where C_1 is the graph of the function $x \mapsto 4 - x^2$ with domain $[-2, 2]$, and C_2 is the graph of the function $x \mapsto x^2 - 4$ with domain $[-2, 2]$. Compute the flux out of the region enclosed by C (this means that the normals are pointing outward) of the vector field $\mathbf{F}(x, y) = (x + y,\ y - x)$.



Solution. For C_1:

Parametrize the curve C_1 by $t \mapsto (t, 4 - t^2),\ t \in [-2, 2]$. The velocity of the parametrization is $\mathbf{v}(t) = (1, -2t)$, and the outward normal is $\mathbf{n}_-(t) = (-y'(x), x'(t)) = (2t, 1)$. Therefore,

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}_-(t) &= \big(\ t + 4 - t^2,\ 4 - t^2 - t\ \big) \cdot (2t, 1) \\
&= 2t^2 + 8t - 2t^3 + 4 - t^2 - t \\
&= -2t^3 + t^2 + 7t + 4,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds &= \int_{-2}^{2} -2t^3 + t^2 + 7t + 4\, dt \\
&= \left[-\frac{t^4}{2} + \frac{t^3}{3} + \frac{7t^2}{2} + 4t \right]_{t=-2}^{t=2} \\
&= \frac{64}{3}.
\end{aligned}$$

Across C_2, we see that the flux must be equal to that across C_1 by symmetry (the set-up is the same if we rotate everything by 180 degrees), so that

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \int_{C_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds + \int_{C_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \frac{64}{3} + \frac{64}{3} = \frac{128}{3}.$$

For the symmetry-argument skeptic, we can check that indeed the flux across C_2 is what we claim:

Parametrize the curve C_2 by $t \mapsto (t,\ t^2 - 4),\ t \in [-2, 2]$. The velocity of the parametrization is $\mathbf{v}(t) = (1, 2t)$, and the outward normal is $\mathbf{n}_+(t) = (y'(t), -x'(t)) = (2t, -1)$. Then,

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) &= \big(\ t + t^2 - 4,\ t^2 - 4 - t\ \big) \cdot (2t, -1) \\
&= 2t^2 + 2t^3 - 8t - t^2 + 4 + t \\
&= 2t^3 + t^2 - 7t + 4
\end{aligned}$$

and so

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds &= \int_{-2}^{2} 2t^3 + t^2 - 7t + 4\,dt\\ &= \left[\frac{t^4}{2} + \frac{t^3}{3} - \frac{7t^2}{2} + 4t\right]_{t=-2}^{t=2}\\ &= \frac{64}{3}.\end{aligned}$$

4. Let C be a simple oriented curve (reminder: this means that C has no self-intersections (except for the possibility that the two endpoints of C may meet), and that one of the two possible orientations of C has been chosen). Denote by $-C$ the simple oriented curve obtained by reversing the orientation of C.

If $t \mapsto \mathbf{r}(t),\ t \in [a,b]$ is a parametrization of C, then one possible parametrization of $-C$ is $t \mapsto \mathbf{r}(b+a-t),\ t \in [a,b]$.

(a) Let L be the line segment in $\mathbb{R}^2$ going from the origin $(0,0)$ to the point $(1,2)$. Parametrize L, denoting your parametrization by $t \mapsto \mathbf{r}(t)$, $t \in [a,b]$. Then, convince yourself that the above prescription produces a parametrization of $-L$: check that $t \mapsto \mathbf{r}(b+a-t),\ t \in [a,b]$ goes from $(1,2)$ to $(0,0)$, and compare its velocity vector with that of your parametrization of L.

(b) Let $\mathbf{F}$ be a vector field. Let C be a simple oriented curve, parametrized as $t \mapsto \mathbf{r}(t)$, with $t \in [a,b]$. Denote the reversed-orientation parametrization above by $t \mapsto \mathbf{s}(t) := \mathbf{r}(b+a-t)$, with $t \in [a,b]$. Show that

$$\int_a^b \mathbf{F}(\mathbf{s}(t))\cdot\mathbf{s}'(t)\,dt = -\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.$$

(*Suggestion:* Expand out in coordinates and make a u-substitution.)

In fact, it is true in general that $\int_{-C}\mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$ (and the proof is similar to the computation above, but involves a discussion of orientation-reversing reparametrizations, which we will omit).

As discussed in lecture, the work integral over a piecewise curve $D = D_1 + D_2 + \cdots + D_n$, where each D_i is simple and oriented, is defined as

$$\int_D \mathbf{F}\cdot d\mathbf{r} := \int_{D_1}\mathbf{F}\cdot d\mathbf{r} + \int_{D_2}\mathbf{F}\cdot d\mathbf{r} + \cdots + \int_{D_n}\mathbf{F}\cdot d\mathbf{r}.$$

For simplicity of notation, write $C + (-C)$ as $C - C$.

(c) Conclude that $\int_{C-C}\mathbf{F}\cdot d\mathbf{r} = 0$.

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Recall the following definition from lecture:

Definition. A vector field $\mathbf{F}$ is called *conservative* if, for any pair of points $Q,\ P$, and any pair of piecewise curves $C,\ C'$ from Q to P, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r}$.

Negating the definition, a vector field $\mathbf{F}$ is *not* conservative if there exists *some* pair of points $Q,\ P$, and *some* pair of piecewise curves $C,\ C'$ from Q to P with $\int_C \mathbf{F}\cdot d\mathbf{r} \neq \int_{C'} \mathbf{F}\cdot d\mathbf{r}$.

(d) Show the stronger statement: if a vector field is not conservative, then for *all* pairs of points $Q,\ P$, there exists some pair of piecewise curves $C,\ C'$ from Q to P with $\int_C \mathbf{F}\cdot d\mathbf{r} \neq \int_{C'} \mathbf{F}\cdot d\mathbf{r}$. (*Suggestion:* Extend by two line segments the two piecewise curves guaranteed by the above negation of the definition of conservative.)

Solution.

(a) L may be parametrized as $t \mapsto (t, 2t)$ with $t \in [0,1]$. We have $a = 0$, $b = 1$, so $b+a-t = 1-t$. Let $\mathbf{s}(t) = \mathbf{r}(b+a-t) = \mathbf{r}(1-t) = (1-t, 2-2t)$.

At $t = 0$, $\mathbf{s}(0) = (1-0, 2-0) = (1,2)$. At $t = 1$, $\mathbf{s}(1) = (1-1, 2-2) = (0,0)$. We have

$$\mathbf{s}'(t) = (-1,-2) = -(1,2) = -\mathbf{r}'(t),$$

so the velocity vectors of the two paths are opposite at every point.

(b) Write $\mathbf{r}(t)$ out in (x,y) coordinates as $\mathbf{r}(t) = (x(t), y(t))$. By definition of $\mathbf{s}$, we have $\mathbf{s}(t) = \mathbf{r}(b+a-t) = (x(b+a-t), y(b+a-t))$.

Since (by the chain rule)

$$\frac{d}{dt}\left(x(b+a-t)\right) = x'(b+a-t)\cdot \frac{d}{dt}(b+a-t) = x'(b+a-t)\cdot(-1) = -x'(b+a-t),$$

and similarly $\frac{d}{dt}\left(y(b+a-t)\right) = -y'(b+a-t)$, we have

$$\begin{aligned}\mathbf{s}'(t) &= \left(\frac{d}{dt}\left(x(b+a-t)\right),\ \frac{d}{dt}\left(y(b+a-t)\right)\right)\\ &= -\left(x'(b+a-t),\ y'(b+a-t)\right)\\ &= -\mathbf{r}'(b+a-t).\end{aligned}$$

Therefore,

$$\int_a^b \mathbf{F}(\mathbf{s}(t))\cdot\mathbf{s}'(t)\,dt = \int_a^b \mathbf{F}(\mathbf{r}(b+a-t))\cdot(-\mathbf{r}'(b+a-t))\,dt$$

Now, make the u-substitution $u = b+a-t,\ du = -dt$. Since t goes from a to b, u goes from $b+a-a = b$ to $b+a-b = a$, and we get

$$\int_a^b \mathbf{F}(\mathbf{s}(t))\cdot\mathbf{s}'(t) = \int_b^a \mathbf{F}(\mathbf{r}(u))\cdot\mathbf{r}'(u)\,du = -\int_a^b \mathbf{F}(\mathbf{r}(u))\cdot\mathbf{r}'(u)\,du,$$

which (up to renaming u by t on the right-hand side) is what we were asked to prove.

(c) $\int_{C-C} \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_{-C} \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} - \int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

(d) Let Q' and P' be a pair of points, and D, D' pair of paths from Q' to P', such that $\int_D \mathbf{F} \cdot d\mathbf{r} \neq \int_{D'} \mathbf{F} \cdot d\mathbf{r}$. Such points and paths exist, because $\mathbf{F}$ is assumed to be not conservative.

Now, let Q and P be an arbitrary pair of points. Let E be a path from Q to Q', and let E' be a path from P' to P (for example, straight line segments joining these pairs of points would do). Then, $C := E + D + E'$ and $C' := E + D' + E'$ is a pair of paths from Q to P, and $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$.

Indeed, by definition,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_E \mathbf{F} \cdot d\mathbf{r} + \int_D \mathbf{F} \cdot d\mathbf{r} + \int_{E'} \mathbf{F} \cdot d\mathbf{r}, \quad \text{and}$$
$$\int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_E \mathbf{F} \cdot d\mathbf{r} + \int_{D'} \mathbf{F} \cdot d\mathbf{r} + \int_{E'} \mathbf{F} \cdot d\mathbf{r}.$$

Since $\int_D \mathbf{F} \cdot d\mathbf{r} \neq \int_{D'} \mathbf{F} \cdot d\mathbf{r}$ by the choice of D and D', and the other two terms are shared between $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$.