MTHE 227 Problem Set 3 Solutions

1. Let f be the function $f(x,y) = x^2y + y$. Define the following paths in $\mathbb{R}^2$:

$$\begin{aligned}
&C_1\colon \text{The line segment} \quad t \mapsto (0, t), \quad t \in [-1, 1]\\
&C_2\colon \text{The sideways parabola segment} \quad t \mapsto (1-t^2, t), \quad t \in [-1, 1]\\
&C_3\colon \text{The left unit semicircle} \quad t \mapsto (-\cos t, \sin t), \quad t \in [-\pi/2, \pi/2].
\end{aligned}$$

Each of the paths connects the points $(0,-1)$ and $(0,1)$.

(a) Compute $f(0,1) - f(0,-1)$.

(b) Let $\mathbf{F} = \nabla f$ be the gradient field of f. Compute $\mathbf{F}$.

(c) Compute the work $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, and $\int_{C_3} \mathbf{F}\cdot d\mathbf{r}$ done by $\mathbf{F}$ along each of the C_i.

(d) Explain the connection between (a) and (c).

Now define the path

$$C_4\colon \text{The line segment} \quad t \mapsto (0, -t), \quad t \in [-1, 1]$$

(e) Compute $\int_{C_4} \mathbf{F}\cdot d\mathbf{r}$, compare with $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, and explain the difference.

Solution. (a) $f(0,1) - f(0,-1) = (0^2(1) + 1) - (0^2(-1) - 1) = 2$.

(b) We have

$$\mathbf{F}(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y)\right) = (2xy,\ x^2+1). \quad \text{for all } (x,y) \in \mathbb{R}^2$$

(c) For C_1:

$$\begin{aligned}
&\mathbf{r}(t) = (0,t),\\
&\mathbf{v}(t) = (0,1),\\
&\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t) = (2\cdot 0\cdot t,\ 0^2+1)\cdot(0,1) = 1,\\
&\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t)\,dt = \int_{-1}^{1} 1\,dt = [t]_{t=-1}^{t=1} = 1-(-1) = 2.
\end{aligned}$$

For C_2:

$$\begin{aligned}
&\mathbf{r}(t) = (1-t^2, t),\\
&\mathbf{v}(t) = (-2t, 1),\\
&\mathbf{F}(\mathbf{r}(t)) = (2\cdot(1-t^2)\cdot t,\ (1-t^2)^2+1) = (2t-2t^3,\ 2-2t^2+t^4),\\
&\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t) = (2t-2t^3,\ 2-2t^2+t^4)\cdot(-2t,1) = (4t^4-4t^2) + (2-2t^2+t^4) = 2-6t^2+5t^4,\\
&\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} 2-6t^2+5t^4\,dt = \left[2t-2t^3+t^5\right]_{t=-1}^{t=1} = (2-2+1) - ((-2)-(-2)+(-1)) = 2.
\end{aligned}$$

For C_3:

$$\begin{aligned}
&\mathbf{r}(t) = (-\cos t, \sin t),\\
&\mathbf{v}(t) = (\sin t, \cos t),\\
&\mathbf{F}(\mathbf{r}(t)) = (2(-\cos t)(\sin t), \cos^2 t + 1) = (-2\cos t \sin t, \cos^2 t + 1),\\
&\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (-2\cos t \sin t, \cos^2 t + 1) \cdot (\sin t, \cos t) = -2\cos t \sin^2 t + \cos^3 t + \cos t,\\
&\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_{-\pi/2}^{\pi/2} -2\cos t \sin^2 t + \cos^3 t + \cos t \, dt.
\end{aligned}$$

Note that $\cos^3 t = \cos^2 t \cos t = (1 - \sin^2 t)\cos t = \cos t - \sin^2 t \cos t$, so the integral is equal to

$$\int_{-\pi/2}^{\pi/2} -3\cos t \sin^2 t + 2\cos t \, dt$$

Split this into two integrals. For the first, let $u = \sin t$, $du = \cos t \, dt$. Then

$$\int_{-\pi/2}^{\pi/2} -3\cos t \sin^2 t \, dt = \int_{-1}^{1} -3u^2 \, du = \left[-u^3\right]_{u=-1}^{u=1} = -1 - 1 = -2.$$

The second is

$$\int_{-\pi/2}^{\pi/2} 2\cos t \, dt = [2\sin t]_{t=-\pi/2}^{t=\pi/2} = 2(1) - 2(-1) = 4.$$

Thus,

$$\int_{C_3} \mathbf{F} \cdot \mathbf{dr} = -2 + 4 = 2.$$

(d) The work done along each C_i is equal to 2, which is of course also equal to the result of part (a). The reason is the Fundamental Theorem of Calculus for Line Integrals — if q and p are two points and C is any oriented path from q to p, then

$$\int_C \nabla f \cdot \mathbf{dr} = f(p) - f(q).$$

(e) For C_4:

$$\begin{aligned}
&\mathbf{r}(t) = (0, -t),\\
&\mathbf{v}(t) = (0, -1),\\
&\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (2 \cdot 0 \cdot (-t), 0^2 + 1) \cdot (0, -1) = -1,\\
&\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) \, dt = \int_{-1}^{1} -1 \, dt = [-t]_{t=-1}^{t=1} = -1 - 1 = -2.
\end{aligned}$$

We see that $\int_{C_4} \mathbf{F} \cdot \mathbf{dr} = -\int_{C_1} \mathbf{F} \cdot \mathbf{dr}$. The sign of work changes when we reverse the orientation of the curve.

2. Let C be upper unit semicircle in $\mathbb{R}^2$, oriented clockwise. Define the following vector fields:

$$\mathbf{F}(x,y) := (x,y), \qquad \mathbf{G}(x,y) := \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right), \qquad \mathbf{H}(x,y) := (-y,x) \qquad \text{for all } (x,y) \in \mathbb{R}^2$$

These three vector fields can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ counterclockwise by $0, \pi/4$ and $\pi/2$ radians, respectively.

Compute $\int_C \mathbf{F} \cdot \mathbf{dr}$, $\int_C \mathbf{G} \cdot \mathbf{dr}$ and $\int_C \mathbf{H} \cdot \mathbf{dr}$. Which of the three is largest? Which is smallest? Explain briefly. (Be careful to parametrize C with the correct orientation.)

Optional Problem. The three vector fields above are members of the family

$$\mathbf{F}_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta),$$

with $0 \le \theta < 2\pi$ ($\mathbf{F} = \mathbf{F}_0$, $\mathbf{G} = \mathbf{F}_{\pi/4}$, $\mathbf{H} = \mathbf{F}_{\pi/2}$). The vector field $\mathbf{F}_\theta$ can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ counterclockwise by θ radians. Plot the work done by $\mathbf{F}_\theta$ along C, as a function of θ.

Solution. Parametrize C as $t \mapsto (-\cos t, \sin t) =: \mathbf{r}(t)$, $t \in [0,\pi]$. The velocity of the parametrization is $\mathbf{v}(t) = (\sin t, \cos t)$. Then,

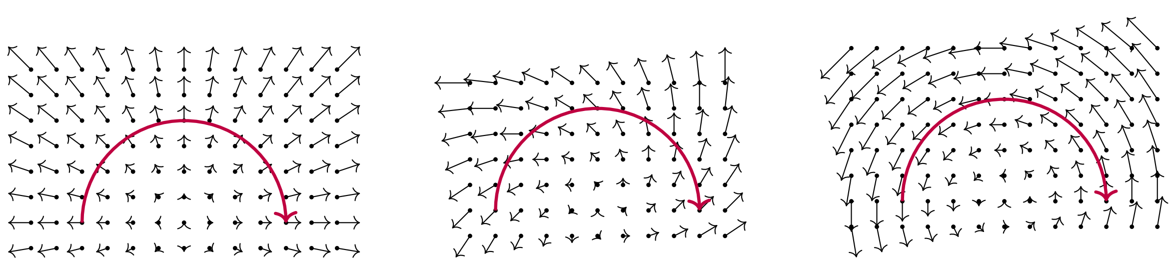
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (-\cos t, \sin t) \cdot (\sin t, \cos t) = -\cos t \sin t + \sin t \cos t = 0,$$

$$\mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = \left(\frac{-\cos t - \sin t}{\sqrt{2}}, \frac{-\cos t + \sin t}{\sqrt{2}}\right) \cdot (\sin t, \cos t) = \frac{-\cos t \sin t - \sin^2 t - \cos^2 t + \sin t \cos t}{\sqrt{2}} = -\frac{1}{\sqrt{2}},$$

$$\mathbf{H}(\mathbf{r}(t)) \cdot \mathbf{v}(t) = (-\sin t, -\cos t) \cdot (\sin t, \cos t) = -\sin^2 t - \cos^2 t = -1.$$

Thus, the work done by $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ along C is

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_0^\pi 0\, dt = 0,$$
$$\int_C \mathbf{G} \cdot \mathbf{dr} = \int_0^\pi -\frac{1}{\sqrt{2}}\, dt = -\frac{\pi}{\sqrt{2}},$$
$$\int_C \mathbf{H} \cdot \mathbf{dr} = \int_0^\pi -1\, dt = -\pi.$$



Here are geometric arguments for why the work done by the three vector fields has to have the values computed above.

First, the position vector field $\mathbf{F}$ is perpendicular to C at every point, and therefore does no work on a particle moving along C.

Second, every vector of $\mathbf{G}$ is a vector of $\mathbf{F}$ rotated by $\pi/4$ radians (=45 degrees) counterclockwise, and therefore makes an angle of $\pi/4$ with the tangent line to C at every point. It follows that the tangential component of $\mathbf{G}$ is precisely $\cos(\pi/4) = 1/\sqrt{2}$. Because $\mathbf{G}$ is directed opposite to the orientation of C, it does negative work, and so the total work done by $\mathbf{G}$ is equal to

$$-\cos(\pi/4)\cdot\text{distance travelled along } C = -(1/\sqrt{2})\pi,$$

since the arclength of a semicircle of radius 1 is equal to π.

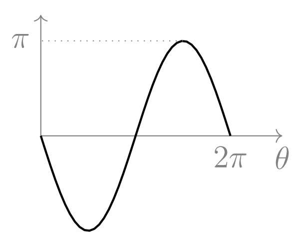
Similarly and lastly, every vector of $\mathbf{H}$ is a vector of $\mathbf{F}$ rotated by $\pi/2$ radians (=90 degrees), and so $\mathbf{H}$ is tangent to C at every point. It follows that the total work done by $\mathbf{H}$ along C is equal to

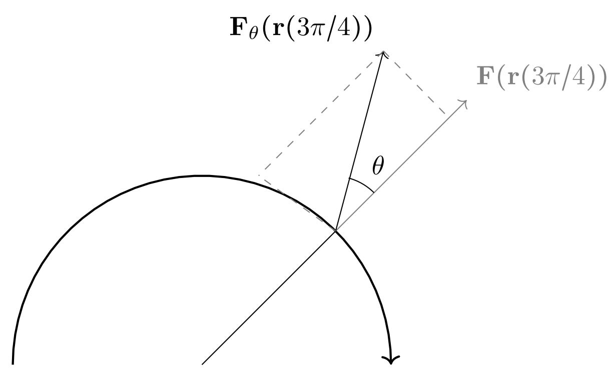
$$-1\cdot\text{distance travelled along } C = -1\pi.$$

Solution of the Optional Problem. We have

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t) &= (-\cos t\cos\theta - \sin t\sin\theta,\ -\cos t\sin\theta + \sin t\cos\theta)\cdot(\sin t,\cos t)\\ &= -\cos t\sin t\cos\theta - \sin^2 t\sin\theta - \cos^2 t\sin\theta + \cos t\sin t\cos\theta\\ &= -(\sin^2 t + \cos^2 t)\sin\theta\\ &= -\sin\theta.\end{aligned}$$

Therefore, the work done by $\mathbf{F}_\theta$ along C is $\int_C \mathbf{F}_\theta\cdot d\mathbf{r} = \int_0^\pi -\sin\theta\,dt = -\sin(\theta)\,\pi$. The graph of work done by $\mathbf{F}_\theta$ along C as a function of θ looks as follows:





Can we see a geometric reason for the result? Since every vector of $\mathbf{F}_\theta$ is a vector of $\mathbf{F}$ rotated by θ radians, the length of the component of $\mathbf{F}_\theta(\mathbf{r}(t))$ that is tangent to C is equal to

$$\|\mathbf{F}_\theta(\mathbf{r}(t))\|\sin\theta = \|\mathbf{F}(\mathbf{r}(t))\|\sin\theta = 1\cdot\sin\theta = \sin\theta,$$

where the first equality is true because rotation does not change the length of a vector, and the second equality is true because $\|\mathbf{F}(x,y)\| = \sqrt{x^2+y^2} = 1$ for every point (x,y) on C. Because the rotation goes counter to the orientation of C, this component goes against the motion, accounting for the minus sign in the work done. This is true at every point of C, so the work done is $-\sin\theta$ times the distance travelled, or $-\sin(\theta)\pi$.

3 (Potential for a 2D Spring)**.** Suppose we have a spring in $\mathbb{R}^2$ whose length at rest is equal to ℓ, with one end of the spring attached to a fixed point. Choose a system of coordinates so that the fixed point is at the origin. Suppose that the spring can be rotated freely (without friction) about the pivot at the origin. If the other end of the spring is moved to position $\mathbf{r} \neq \vec{0}$ in $\mathbb{R}^2$, *Hooke's Law* tells us that the force exerted by the spring on a particle (of unit mass) attached to the non-pivoted end is equal to

$$\mathbf{F}(\mathbf{r}) = -k(\|\mathbf{r}\| - \ell)\,\frac{\mathbf{r}}{\|\mathbf{r}\|}, \tag{1}$$

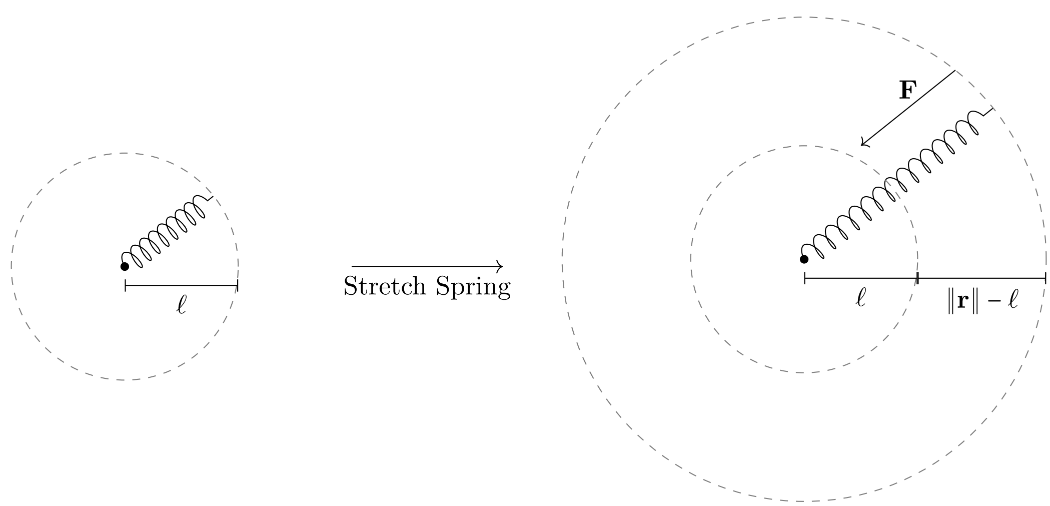
where $k > 0$ is a constant. In other words, the force is proportional to the distance the spring is moved from its resting position, and directed radially toward the resting position.[1] (It is sometimes called a *restoring force.*)

(a) Check that in (x, y)-coordinates expression (1) becomes

$$\mathbf{F}(x, y) = \left(-kx + \frac{k\ell x}{\sqrt{x^2 + y^2}},\ -ky + \frac{k\ell y}{\sqrt{x^2 + y^2}}\right).$$

(b) Find a potential φ for this vector field, normalized so that $\varphi(\ell, 0) = 0$. Show that the potential can be written as a function of $r := \sqrt{x^2 + y^2}$ only. What do the equipotential curves look like?

(c) Find the work required to rotate and stretch the spring from the point $(\ell, 0)$ to the point $(0, R)$, for any $R \geq \ell$.



[1] This is an empirical law that approximates real springs fairly well, as long as $\|\mathbf{r}\|$ is relatively close to ℓ.

Solution.

(a) Since $\mathbf{r}(x,y) = (x,y)$ and $\|\mathbf{r}(x,y)\| = \sqrt{x^2+y^2}$, we have

$$\begin{aligned}
-k(\|\mathbf{r}\|-\ell)\frac{\mathbf{r}}{\|\mathbf{r}\|} &= -k(\sqrt{x^2+y^2}-\ell)\frac{(x,y)}{\sqrt{x^2+y^2}} \\
&= -k(x,y) + k\ell\frac{(x,y)}{\sqrt{x^2+y^2}} \\
&= (-kx,-ky) + \left(\frac{k\ell x}{\sqrt{x^2+y^2}}, \frac{k\ell y}{\sqrt{x^2+y^2}}\right) \\
&= \left(-kx + \frac{k\ell x}{\sqrt{x^2+y^2}}, -ky + \frac{k\ell y}{\sqrt{x^2+y^2}}\right).
\end{aligned}$$

(b) We use Method 2 from class, but any other method would also be acceptable (for instance, one could guess based on knowledge of the potential for linear springs from the one-dimensional case).

If ϕ exists, we must have $\frac{\partial\phi}{\partial x} = -kx + \frac{k\ell x}{\sqrt{x^2+y^2}}$. Integrating both sides with respect to x, we get

$$\phi(x,y) = -\frac{k}{2}x^2 + k\ell\int\frac{x}{\sqrt{x^2+y^2}}\,dx + g(y),$$

for some function $g(y)$ of y only. The integral can be done using the u-substitution $u = x^2+y^2$, $du = 2x\,dx$ —

$$\int\frac{x}{\sqrt{x^2+y^2}}\,dx = \int\frac{1}{2}\frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{x^2+y^2}$$

(the usual arbitrary constant can be absorbed into the $g(y)$ term). Putting the previous two expressions together,

$$\phi(x,y) = -\frac{k}{2}x^2 + k\ell\sqrt{x^2+y^2} + g(y).$$

Taking the partial with respect to y, we have

$$\frac{\partial\phi}{\partial y} = \frac{1}{2}\frac{k\ell}{\sqrt{x^2+y^2}}\cdot 2y + g'(y) = \frac{k\ell y}{\sqrt{x^2+y^2}} + g'(y).$$

Comparing with the y-component of $\mathbf{F}$, we see that

$$g'(y) = -ky \implies g(y) = -\frac{k}{2}y^2 + C.$$

So that a general potential has the form

$$\phi(x,y) = -\frac{k}{2}(x^2+y^2) + k\ell\sqrt{x^2+y^2} + C.$$

From the requirement $\phi(\ell, 0) = 0$, we get

$$0 = \phi(\ell, 0) = -\frac{k}{2}(\ell^2 + 0^2) + k\ell\sqrt{\ell^2 + 0^2} + C = -\frac{k}{2}\ell^2 + k\ell^2 + C,$$

so that

$$C = -\frac{k}{2}\ell^2$$

and

$$\phi(x, y) = -\frac{k}{2}(x^2 + y^2) + k\ell\sqrt{x^2 + y^2} - \frac{k}{2}\ell^2 = -\frac{k}{2}\left(x^2 + y^2 - 2\ell\sqrt{x^2 + y^2} + \ell^2\right).$$

Rewriting the above expression in terms of $r = \sqrt{x^2 + y^2}$, we get

$$\phi(r) = -\frac{k}{2}\left(r^2 - 2\ell r + \ell^2\right) = -\frac{k}{2}\left(r - \ell\right)^2.$$

Notice that this is very similar to the potential $-\frac{k}{2}(x - \ell)^2$ for a linear spring in one dimension! Since $r(x, y) = \sqrt{x^2 + y^2}$ is equal to the distance of the point (x, y) from the origin, we see that the equipotential curves are circles centered at the origin. The potential is 0 for $r = \ell$ (when the spring is at rest), and decreases on circles, as the spring is either compressed or stretched.

(c) To find the work the spring does along any path C from $(\ell, 0)$ to $(0, R)$, we can simply find the potential difference. Since $r(0, R) = \sqrt{0^2 + R^2} = R$ and $r(\ell, 0) = \ell$,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \nabla\phi \cdot \mathbf{dr} = \phi(0, R) - \phi(\ell, 0) = -\frac{k}{2}(R - \ell)^2 - \left(-\frac{k}{2}(\ell - \ell)^2\right) = -\frac{k}{2}(R - \ell)^2.$$

To stretch the spring (with minimal possible work), we need to apply a force equal and opposite to that of the spring, so that the work required to stretch the spring is

$$\frac{k}{2}(R - \ell)^2.$$