MTHE 227 Problem Set 2 Solutions

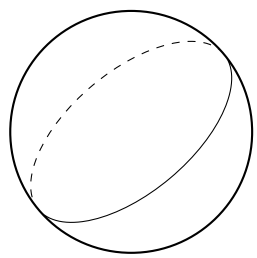
1 (Great Circles)**.** The intersection of a sphere with a plane passing through its center is called a *great circle*. Let Γ be the great circle that is the intersection of the plane $x+y+z=0$ with the sphere $x^2+y^2+z^2=R^2$ of radius R centered at the origin.

(a) Find a parametrization of Γ. (*Suggestion:* Begin by finding two perpendicular vectors lying in the plane $x+y+z=0$.)

(b) Check that the arclength of Γ is equal to $2\pi R$.

(c) If $t \mapsto (x(t), y(t), z(t))$ is a parametrization of Γ, explain why $t \mapsto (x(t), y(t), -z(t))$ is a parametrization of the great circle Γ' cut out from the same sphere by $x+y-z=0$.

Optional Problem. Find the points of intersection of Γ and Γ', and find the angle of intersection between the tangent lines to Γ and Γ' at these points.



Solution. (a) Following the suggestion, we begin by finding a pair of perpendicular vectors that lie in the plane $x+y+z=0$.

Take $\mathbf{v_1} = (1,-1,0)$. Since $1+(-1)+0=0$, $\mathbf{v_1}$ lies in the plane. Let $\mathbf{v_2} = (a,b,c)$ be a vector in the same plane that is perpendicular to $\mathbf{v_1}$. The coordinates a,b,c must satisfy the system of linear equations

$$\begin{aligned} a+b+c &= 0, \\ a-b &= 0, \end{aligned}$$

where the second equation comes from $\mathbf{v_1}\cdot\mathbf{v_2}=0$. We see (using Gauss-Jordan elimination from Linear Algebra, or doing it by hand) that all solutions are of the form $b=a$ and $c=-2a$, or $(a,a,-2a)$, with a some arbitrary real number. A particular solution is $\mathbf{v_2}=(1,1,-2)$ (taking $a=1$).

Now, rescale $\mathbf{v_1}$ and $\mathbf{v_2}$ so that the two vectors lie on the sphere $x^2+y^2+z^2=R^2$. For this to hold, their lengths $\|\mathbf{v_1}\|$ and $\|\mathbf{v_2}\|$ must be equal to R.

We have $\|\mathbf{v_1}\| = \sqrt{1^2+(-1)^2+0^2} = \sqrt{2}$, so

$$\frac{R}{\sqrt{2}}\mathbf{v_1} = \left(\frac{R}{\sqrt{2}}, -\frac{R}{\sqrt{2}}, 0\right)$$

lies on the surface of the sphere. Denote the vector $\frac{R}{\sqrt{2}}\mathbf{v_1}$ by $\mathbf{w_1}$.

Similarly, $\|\mathbf{v_2}\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$, so

$$\frac{R}{\sqrt{6}}\mathbf{v_2} = \left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, \frac{-2R}{\sqrt{6}}\right)$$

lies on the surface of the sphere, and is perpendicular to $\mathbf{w_1}$. Denote the vector $\frac{R}{\sqrt{6}}\mathbf{v_2}$ by $\mathbf{w_2}$.

We have constructed two perpendicular vectors that lie on the great circle Γ. If $\mathbf{w}$ is the direction vector of an arbitrary point on the great circle, we can express it 'in $\mathbf{w_1}$, $\mathbf{w_2}$-coordinates' as $\mathbf{w_1}\cos\theta + \mathbf{w_2}\sin\theta$, where θ is the counterclockwise angle that $\mathbf{w}$ makes with $\mathbf{w_1}$ (think about the projections of $\mathbf{w}$ onto $\mathbf{w_1}$ and $\mathbf{w_2}$). Thus, Γ can be parametrized as

$$\theta \mapsto \mathbf{w_1}\cos\theta + \mathbf{w_2}\sin\theta = \frac{R}{\sqrt{6}}\left(\sin\theta + \sqrt{3}\cos\theta,\ \sin\theta - \sqrt{3}\cos\theta,\ -2\sin\theta\right), \quad \theta \in [0, 2\pi).$$

(b) The velocity of the parametrization of part (a) is

$$\mathbf{v}(\theta) = \frac{R}{\sqrt{6}}\left(\cos\theta - \sqrt{3}\sin\theta,\ \cos\theta + \sqrt{3}\sin\theta,\ -2\cos\theta\right).$$

Therefore, the square of the speed is

$$\begin{aligned}
\|\mathbf{v}(\theta)\|^2 &= \frac{R^2}{6}\left((\cos\theta - \sqrt{3}\sin\theta)^2 + (\cos\theta + \sqrt{3}\sin\theta)^2 + (-2\cos\theta)^2\right) \\
&= \frac{R^2}{6}\left((\cos^2\theta - 2\sqrt{3}\cos\theta\sin\theta + 3\sin^2\theta) + (\cos^2\theta + 2\sqrt{3}\cos\theta\sin\theta + 3\sin^2\theta) + 4\cos^2\theta\right) \\
&= \frac{R^2}{6}\left(6\cos^2\theta + 6\sin^2\theta\right) \\
&= \frac{R^2}{6}6 \\
&= R^2.
\end{aligned}$$

We conclude that $\|\mathbf{v}(\theta)\| = R$.

The arclength of Γ is then

$$\int_0^{2\pi} \|\mathbf{v}(\theta)\|\,d\theta = \int_0^{2\pi} R\,d\theta = R\int_0^{2\pi} d\theta = 2\pi R.$$

We might have expected this result, as, after all, Γ is a circle of radius R in the plane $x + y + z = 0$, and so should have circumference $2\pi R$!

(c) If $t \mapsto (x(t), y(t), z(t))$ is a parametrization of Γ, every point of the type $(x(t), y(t), -z(t))$ satisfies $x^2 + y^2 + z^2 = R^2$ and $x + y - z = 0$. So, every point of the path $t \mapsto (x(t), y(t), -z(t))$ lies on Γ'. Conversely, every point of Γ' has a corresponding point on Γ (the point $(x, y, -z)$ on Γ is sent to (x, y, z) on Γ'), so every point of Γ' is included in the path.

The curve Γ' is a reflection of Γ in the xy-plane.

Solution to the Optional Problem. The great circles Γ and Γ' are parametrized by

$$\Gamma\colon \theta \mapsto \frac{R}{\sqrt{6}}\left(\sin\theta + \sqrt{3}\cos\theta,\, \sin\theta - \sqrt{3}\cos\theta,\, -2\sin\theta\right), \quad \theta \in [0, 2\pi),$$
$$\Gamma'\colon \phi \mapsto \frac{R}{\sqrt{6}}\left(\sin\phi + \sqrt{3}\cos\phi,\, \sin\phi - \sqrt{3}\cos\phi,\, 2\sin\phi\right), \quad \phi \in [0, 2\pi).$$

The great circles intersect when their coordinates are equal. When this happens, looking at the third coordinate, we must have

$$-2\sin\theta = 2\sin\phi,$$

so that $\sin\theta = -\sin\phi$, which implies that either $\theta = \phi \pm \pi$ or that both θ and ϕ are elements of the set $\{0, \pi\}$.

Considering the various cases in turn (some details are omitted here), we find that there are two possibilities: $\theta = \phi = 0$ and $\theta = \phi = \pi$. Alternatively, as Γ' is the reflection of Γ in the xy-plane, the only points Γ and Γ' have in common are those that lie in the xy-plane (these are the only points fixed by the reflection). This requires that $-2\sin\theta = 2\sin\phi = 0$, and again gives the above solutions (strictly speaking, we also need to check that $\theta = 0$, $\phi = \pi$ and $\theta = \pi$, $\phi = 0$ are not intersection points).

The points of intersection of Γ and Γ' are then

$$p = \left(\frac{R}{\sqrt{2}}, -\frac{R}{\sqrt{2}}, 0\right) \quad \text{and} \quad q = \left(-\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, 0\right).$$

The velocity of the path parametrizing Γ at $t = 0$ is $\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, -\sqrt{\frac{2}{3}}R\right)$. The velocity of the path parametrizing Γ' at $t = 0$ is $\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, \sqrt{\frac{2}{3}}R\right)$. These vectors are the direction vectors of the respective tangent lines, so the angle between the two tangent lines is equal to the angle between the two velocity vectors.

We compute

$$\frac{\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, -\sqrt{\frac{2}{3}}R\right)\cdot\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, \sqrt{\frac{2}{3}}R\right)}{\left\|\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, -\sqrt{\frac{2}{3}}R\right)\right\|\left\|\left(\frac{R}{\sqrt{6}}, \frac{R}{\sqrt{6}}, \sqrt{\frac{2}{3}}R\right)\right\|} = \frac{\frac{R^2}{6} + \frac{R^2}{6} - \frac{2R^2}{3}}{\left(\frac{R^2}{6} + \frac{R^2}{6} + \frac{2R^2}{3}\right)} = \frac{-R^2/3}{R^2} = -1/3.$$

Therefore, the angle of intersection at p is equal to

$$\arccos(-1/3) \approx 1.91\,\text{rad} \approx 109.5^\circ.$$

This is also equal to the angle of intersection at q (by similar computations, or by symmetry).

A slight subtlety: the angle of intersection between two curves is sometimes defined to be the *smaller* of the two angles subtended by their tangent lines at the point of intersection. With this definition, the complementary angle $\pi - \arccos(-1/3) \approx 70.5^\circ$ is the angle of intersection between Γ and Γ'.

2 (Velocity Perpendicular to Position)**.** For a pair of parametrized paths $\mathbf{q}(t)$, $\mathbf{r}(t)$ in $\mathbb{R}^3$, show that

$$\frac{d}{dt}\left(\mathbf{q}(t)\cdot\mathbf{r}(t)\right) = \mathbf{q}'(t)\cdot\mathbf{r}(t) + \mathbf{q}(t)\cdot\mathbf{r}'(t)$$

(here $\cdot$ denotes the dot product and $'$ the derivative with respect to t). Apply this identity to show the following: if the velocity of a parametrized path is always perpendicular to its position, then the curve traced out by the parametrization lies on the surface of a sphere.

Solution. Write out the paths $\mathbf{q}(t)$ and $\mathbf{r}(t)$ in coordinates as

$$\mathbf{q}(t) = (q_1(t), q_2(t), q_3(t)) \qquad \text{and} \qquad \mathbf{r}(t) = (r_1(t), r_2(t), r_3(t)).$$

Their dot product is then

$$\mathbf{q}(t)\cdot\mathbf{r}(t) = q_1(t)r_1(t) + q_2(t)r_2(t) + q_3(t)r_3(t).$$

Differentiating with respect to t and using the product rule, we find that

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{q}(t)\cdot\mathbf{r}(t)\right) &= (q_1'(t)r_1(t) + q_1(t)r_1'(t)) + (q_2'(t)r_2(t) + q_2(t)r_2'(t)) + (q_3'(t)r_3(t) + q_3(t)r_3'(t))\\ &= (q_1'(t)r_1(t) + q_2'(t)r_2(t) + q_3'(t)r_3(t)) + (q_1(t)r_1'(t) + q_2(t)r_2'(t) + q_3(t)r_3'(t))\\ &= \mathbf{q}'(t)\cdot\mathbf{r}(t) + \mathbf{q}(t)\cdot\mathbf{r}'(t),\end{aligned}$$

which proves the identity.

Now, let $\mathbf{r}(t)$ represent a path of a particle. The condition that velocity is always perpendicular to the position is equivalent to the condition that $\mathbf{v}(t)\cdot\mathbf{r}(t) = 0$ for all t. Apply the identity above with $\mathbf{q}(t) = \mathbf{r}(t)$, obtaining

$$\frac{d}{dt}\left(\|\mathbf{r}(t)\|^2\right) = \frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right) = \mathbf{v}(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{v}(t) = 0 + 0 = 0.$$

This implies that $\|\mathbf{r}(t)\|^2$ is a constant, and hence so is $\|\mathbf{r}(t)\|$. In other words, $\mathbf{r}(t)$ lies on the surface of a sphere of radius $\|\mathbf{r}(t)\|$ centered at the origin.

3 (Flow Lines). (a) Check that the path $t \mapsto (2\cos(2t), \sin 2t)$ is a flow line of the vector field $\mathbf{F}(x, y) = (-4y, x)$ on $\mathbb{R}^2$. Sketch the vector field, the path, and check that the path is everywhere tangent to $\mathbf{F}$.

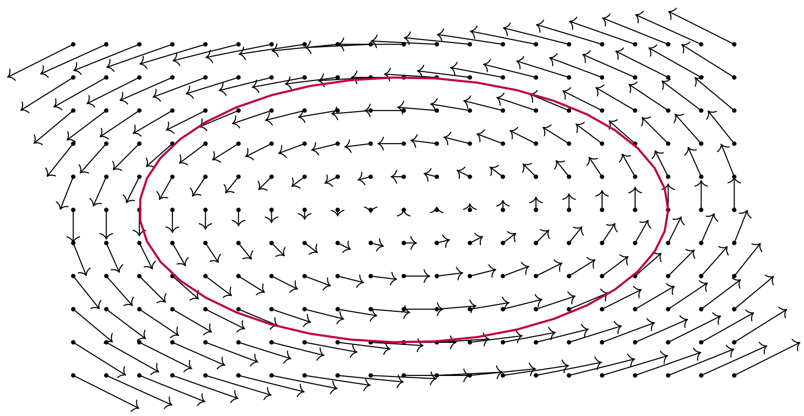
(b) Find the flow lines of the vector field $\mathbf{G}(x, y) = (1, -y^2)$, defined on the first quadrant $\{(x, y) \colon x > 0,\ y > 0\}$ of $\mathbb{R}^2$. Which flow line passes through the point $(1, 1)$? (*Hint:* It may help to find the derivative $\frac{d}{dt}(1/t)$.)

Solution. (a) The velocity of the path at time t is

$$\left(\frac{d}{dt}(2\cos(2t),\ \frac{d}{dt}(\sin(2t)\right) = (-4\sin(2t),\ 2\cos(2t)),$$

which is indeed equal to $\mathbf{F}(2\cos(2t), \sin(2t)) = (-4\sin(2t), 2\cos(2t))$ for all t.

The flow line is an ellipse with semimajor axis of length 2 in the x-direction and semiminor axis of length 1 in the y-direction. A patch of the vector field near the origin looks as follows (with the flow line ellipse traced in bold):



(b) Writing out in coordinates the condition for the parametrized path $t \mapsto (x(t), y(t))$ to be a flow line of the vector field $\mathbf{G}$, we obtain the system of differential equations

$$\begin{aligned} x'(t) &= 1, \\ y'(t) &= -y^2(t). \end{aligned}$$

These are uncoupled equations (that is, $y(t)$ does not appear in the expression for $x'(t)$ and $x(t)$ does not appear in the expression for $y'(t)$), so we can solve the equations separately.

Integrate the first to get $x(t) = t + x_0$, for some undetermined constant x_0.

To solve the second, we could have noticed from the hint that the function $1/t$ is a solution (indeed, $d/dt(1/t) = -1/t^2$), so that an arbitrary solution has the form $1/(t + C)$ for some undetermined constant C. More systematically, the equation is separable, so we separate the variables:

$$\frac{dy}{-y^2} = dt$$

and integrate to get

$$\int \frac{dy}{-y^2} = \frac{1}{y} = \int dt = t + C,$$

for some undetermined constant C, so that

$$y(t) = \frac{1}{t+C}.$$

If $y(0) = y_0$, we get $y_0 = \frac{1}{C}$, so $C = 1/y_0$. The general flow line may be written as

$$(x(t), y(t)) = \left(t + x_0, \frac{1}{t + 1/y_0}\right), \quad t > \max(-1, -1/y_0).$$

In other words, a general flow line is a translated hyperbola.

The flow line that starts at $(x_0, y_0) = (1, 1)$ is then parametrized by $t \mapsto (t+1, 1/(t+1))$, $t > -1$.

A patch of the vector field and some flow lines are plotted below, with the flow line that starts at $(1, 1)$ plotted in blue: