

MTHE 227 PROBLEM SET 1 SOLUTIONS

1 (Secant and Tangent). This problem describes an interpretation of the trigonometric functions \sec and \tan in terms of the geometry of the unit circle.

Let C be the unit circle in \mathbb{R}^2 centered at the origin. Parametrize an arc of C by

$$\begin{cases} x(\theta) = \cos \theta \\ y(\theta) = \sin \theta \end{cases}, \quad -\pi/2 < \theta < \pi/2.$$

- (a) Draw a picture of this arc. Is the arc traversed clockwise or counterclockwise? Is either endpoint included in the path? Find θ so that $(x(\theta), y(\theta)) = (1, 0)$.

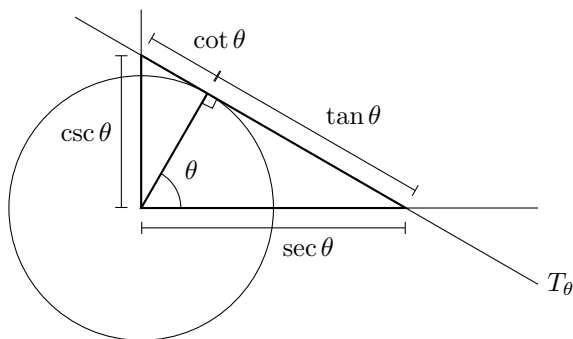
For each $-\pi/2 < \theta < \pi/2$, denote the tangent line to C at $(x(\theta), y(\theta))$ by T_θ , and denote the point of intersection of T_θ with the x -axis by p_θ .

- (b) Parametrize the line T_θ .

Definitions. For each $-\pi/2 < \theta < \pi/2$, define $\sec \theta$ as the distance of p_θ from the origin, and define $\tan \theta$ as the distance of p_θ from the point of tangency of T_θ with C (please see the picture on the bottom left).

- (c) Why can't $\sec(\pi/2)$, $\tan(\pi/2)$, $\sec(-\pi/2)$ and $\tan(-\pi/2)$ be defined similarly?
- (d) Show that $p_\theta = (1/\cos \theta, 0)$, and conclude that $\sec \theta = 1/\cos \theta$ and $\tan \theta = \sin \theta / \cos \theta$. Thus, the above definitions of \sec and \tan are equivalent to the usual analytic ones.¹
- (e) Show that the identity $\tan^2 \theta + 1 = \sec^2 \theta$ holds, in two ways: first, using the above geometric definitions, and second, using the analytic definitions recovered in part (d).

Optional Problem. Show part (d) without using a parametrization.



The cosecant and cotangent have similar interpretations—keeping the notation above and letting q_θ denote the point of intersection of the tangent line T_θ with the y -axis, $\csc \theta$ and $\cot \theta$ are equal to the distance of q_θ from the origin and point of tangency, respectively (the prefix *co* refers to the complementary angle $\pi/2 - \theta$). Although the sine and cosine turn out to have more importance in mathematics, all trigonometric functions are useful, and their geometric meaning makes identities such as

$$(\tan \theta + \cot \theta)^2 = \sec^2 \theta + \csc^2 \theta \text{ and}$$

$$\tan(\pi/2 - \theta) = \cot(\theta)$$

more transparent.

¹At least in the range $-\pi/2 < \theta < \pi/2$, but the definitions extend quite readily to any $\theta \neq n\pi + \pi/2$.

Solution. (a) The parametrization

$$\begin{cases} x(\theta) = \cos \theta \\ y(\theta) = \sin \theta \end{cases}, \quad -\pi/2 < \theta < \pi/2.$$

describes the right unit semicircle, traversed counterclockwise. Neither endpoint is included in the path. The unique value of θ with $(x(\theta), y(\theta)) = (1, 0)$ is $\theta = 0$.



(b) The velocity of the parametrized path of part (a) is equal to

$$\mathbf{v}(\theta) = (-\sin \theta, \cos \theta).$$

Therefore, for a fixed $-\pi/2 < \theta < \pi/2$, the tangent line T_θ is parametrized by

$$t \mapsto \mathbf{r}(\theta) + t\mathbf{v}(\theta) = (\cos \theta, \sin \theta) + t(-\sin \theta, \cos \theta) = (\cos \theta - t \sin \theta, \sin \theta + t \cos \theta), \quad \text{with } t \in \mathbb{R}.$$

(c) The tangent lines to C at $(0, 1) = (\cos(\pi/2), \sin(\pi/2))$ and $(0, -1) = (\cos(-\pi/2), \sin(-\pi/2))$ are parallel to the x -axis, and so do not intersect it. So, p_θ (and hence $\sec \theta$ and $\tan \theta$) cannot be defined in the same way.

(d) The parametrization of T_θ found in part (b) intersects the x -axis when its y -coordinate, $\sin \theta + t \cos \theta$, is equal to 0. This happens at $t = -\sin \theta / \cos \theta$. The corresponding x -coordinate is

$$\cos \theta - \left(-\frac{\sin \theta}{\cos \theta}\right) \sin \theta = \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta} = \frac{1}{\cos \theta}.$$

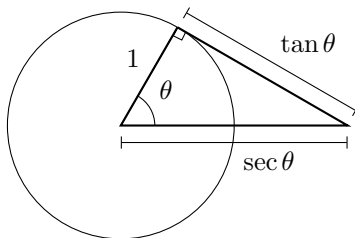
Thus, the point p_θ has coordinates $(1/\cos \theta, 0)$.

It is clear that the distance of p_θ from the origin is $1/\cos \theta$, hence $\sec \theta = 1/\cos \theta$. The point of tangency of T_θ with C is $(\cos \theta, \sin \theta)$, and the distance of this point from p_θ may be found using the distance formula for \mathbb{R}^2 :

$$\begin{aligned} \sqrt{\left(\cos \theta - \frac{1}{\cos \theta}\right)^2 + (\sin \theta - 0)^2} &= \sqrt{\left(\cos^2 \theta - 2 + \frac{1}{\cos^2 \theta}\right) + \sin^2 \theta} \\ &= \sqrt{\frac{1}{\cos^2 \theta} - 1} \\ &= \sqrt{\frac{\cos^2 \theta + \sin^2 \theta - \cos^2 \theta}{\cos^2 \theta}} \\ &= \sqrt{\frac{\sin^2 \theta}{\cos^2 \theta}} \\ &= \frac{\sin \theta}{\cos \theta}. \end{aligned}$$

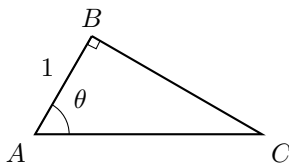
Hence, $\tan \theta = \sin \theta / \cos \theta$.

- (e) Since the unit circle has radius 1, the identity $\tan^2 \theta + 1 = \sec^2 \theta$ comes from applying Pythagoras' theorem to the right triangle below



Analytically, dividing the known identity $\cos^2 \theta + \sin^2 \theta = 1$ by $\cos^2 \theta$, we obtain the desired $1 + \tan^2 \theta = \sec^2 \theta$.

Solution of the Optional Problem. Look again at the right triangle formed by the radius of the circle, the tangent line at $(\cos \theta, \sin \theta)$, and the x -axis, and label its vertices A , B and C as below:



We have

$$\tan \theta = \frac{\text{opp.}}{\text{adj.}} = \frac{\|BC\|}{1} \quad \text{and} \quad \sec \theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{\|AC\|}{1},$$

whence $\|BC\| = \tan \theta$ and $\|AC\| = \sec \theta$.

Much simpler! Perhaps one conclusion is that solving problems using a parametrization is not always the simplest approach.

2 (Intersection of Two Surfaces). The pair of surfaces in \mathbb{R}^3 defined by $x^5 + 4y^2 + z = 3x^2y$ and $5x^3 = y + 6$ intersect along a curve C .

- (a) Parametrize C . (*Suggestion:* For points on C , express y and z as functions of x .)
 (b) Parametrize the tangent line to C at the point $(1, -1, -8)$.

Solution. (a) Since points on C satisfy $5x^3 = y + 6$, their y -coordinate may be expressed as a function of x as

$$y(x) = 5x^3 - 6.$$

Then, since points on C satisfy $x^5 + 4y^2 + z = 3x^2y$, their z -coordinate may be expressed as a function of x as

$$z(x) = 3x^2y - x^5 - 4y^2 = 3x^2(5x^3 - 6) - x^5 - 4(5x^3 - 6)^2 = -100x^6 + 14x^5 + 240x^3 - 18x^2 - 144.$$

Hence, one possible parametrization of C is

$$x \mapsto (x, 5x^3 - 6, -100x^6 + 14x^5 + 240x^3 - 18x^2 - 144), \quad x \in \mathbb{R}.$$

(b) The velocity of the parametrization of part (a) is

$$\mathbf{v}(x) = (1, 15x^2, -600x^5 + 70x^4 + 720x^2 - 36x)$$

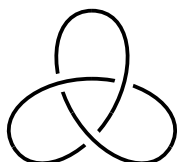
As usual, the tangent line to C at $x = x_0$ may be parametrized by

$$t \mapsto \mathbf{r}(x_0) + t\mathbf{v}(x_0), \quad t \in \mathbb{R}.$$

At $x = 1$, $\mathbf{r}(1) = (1, -1, -8)$ and $\mathbf{v}(1) = (1, 15, 154)$, so the tangent line may be parametrized by

$$t \mapsto (1 + t, -1 + 15t, -8 + 154t), \quad t \in \mathbb{R}.$$

3 (Arclength of the Trefoil Knot). There exist simple closed curves in \mathbb{R}^3 that cannot be continuously deformed to the unit circle without introducing self-intersections along the way. Such curves are called *knotted*, and the simplest knotted curve is known as the *trefoil knot*. One possible parametrization of the trefoil knot² is given by



$$\begin{cases} x(t) = (2 + \cos 3t) \cos 2t \\ y(t) = (2 + \cos 3t) \sin 2t \\ z(t) = \sin 3t \end{cases}, \quad 0 \leq t \leq 2\pi.$$

(a) Verify that the arclength of the trace of the given parametrization is given by

$$\int_0^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt.$$

Unfortunately, this integral is too hard (and perhaps even not possible) to find using methods and functions learned about in first year. Nevertheless, it is possible to understand something about the arclength of the trefoil by bounding the integrand above and below:

(b) Show that the integrand satisfies

$$3 \leq \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} \leq 3\sqrt{5} \quad \text{for all } t.$$

(c) Apply the general estimate from class to conclude that

$$6\pi \leq \text{Arclength of trefoil} \leq 6\pi\sqrt{5}.$$

(These three numbers are close to 20, 30 and 40, respectively.)

Optional Problem. We were not very careful in our analysis, and it is possible to find better bounds. Do this, and tell the instructor about them! For example, can you show that

$$8\pi \leq \text{Arclength of trefoil} \leq (3\sqrt{5} + \sqrt{29})\pi?$$

²More precisely, the *positive* trefoil. The trefoil's mirror image is considered to be a different knot, as it cannot be continuously deformed to the positive trefoil (without introducing self-intersections).

Solution. (a) The three derivatives are (writing $\cos(nt)$ as $\cos nt$ and similarly for \sin to reduce clutter)

$$\begin{aligned}\frac{dx}{dt} &= (-3 \sin 3t) \cos 2t + (2 + \cos 3t)(-2 \sin 2t) = -(3 \sin 3t \cos 2t + 2(2 + \cos 3t) \sin 2t) \\ \frac{dy}{dt} &= (-3 \sin 3t) \sin 2t + (2 + \cos 3t)(2 \cos 2t) = -3 \sin 3t \sin 2t + 2(2 + \cos 3t) \cos 2t \\ \frac{dz}{dt} &= 3 \cos 3t\end{aligned}$$

Squaring the derivatives, we get

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 &= 9 \sin^2 3t \cos^2 2t + 12 \sin 3t \cos 2t(2 + \cos 3t) \sin 2t + 4(2 + \cos 3t)^2 \sin^2 2t \\ \left(\frac{dy}{dt}\right)^2 &= 9 \sin^2 3t \sin^2 2t - 12 \sin 3t \cos 2t(2 + \cos 3t) \sin 2t + 4(2 + \cos 3t)^2 \cos^2 2t \\ \left(\frac{dz}{dt}\right)^2 &= 9 \cos^2 3t\end{aligned}$$

Summing the squares, we get

$$\begin{aligned}9 \sin^2 3t(\cos^2 2t + \sin^2 2t) + 4(2 + \cos 3t)^2(\sin^2 2t + \cos^2 2t) + 9 \cos^2 3t \\ = 9 + 4(2 + \cos 3t)^2 \\ = 25 + 16 \cos(3t) + 4 \cos^2(3t).\end{aligned}$$

This is the function $\|\mathbf{v}(t)\|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$. The arclength integral is then

$$\int_0^{2\pi} \|\mathbf{v}(t)\| dt = \int_0^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt.$$

(b) For all t , we have the bounds

$$-1 \leq \cos(3t) \leq 1 \quad \text{and} \quad 0 \leq \cos^2(3t) \leq 1.$$

Therefore, we have the lower bound

$$\sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} \geq \sqrt{25 - 16 + 0} = \sqrt{9} = 3$$

and the upper bound

$$\sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} \leq \sqrt{25 + 16 + 4} = \sqrt{45} = 3\sqrt{5}.$$

(c) The length of the parametrizing interval is 2π , hence from the general estimates from class we conclude that

$$3 \cdot 2\pi = 6\pi \leq \text{Arclength of trefoil} \leq 3\sqrt{5} \cdot 2\pi = 6\pi\sqrt{5}.$$

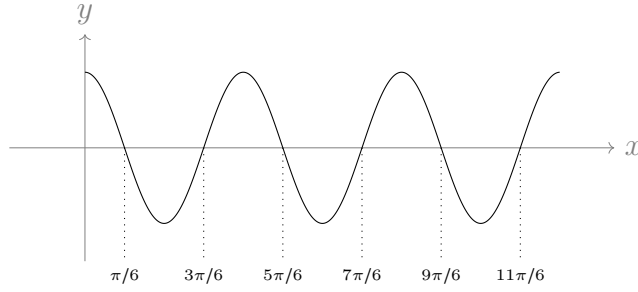
Solution of the Optional Problem. To show the claimed bound, we make use of the following slightly improved bounds on $\cos t$:

$$\begin{aligned} 0 &\leq \cos(t) \leq 1, & t \in [0, \pi/2]; \\ -1 &\leq \cos(t) \leq 0, & t \in [\pi/2, 3\pi/2]; \\ 0 &\leq \cos(t) \leq 1, & t \in [3\pi/2, 2\pi]. \end{aligned}$$

Split the interval $[0, 2\pi]$ into the following pieces, according to whether $\cos(3t)$ is positive or negative over them:

$$[0, \pi/6], [\pi/6, 3\pi/6], [3\pi/6, 5\pi/6], [5\pi/6, 7\pi/6], [7\pi/6, 9\pi/6], [9\pi/6, 11\pi/6], [11\pi/6, 2\pi].$$

Over these pieces, $\cos(3t)$ starts positive, and then alternates between being positive and negative.



The lower bound on $\cos(3t)$ is therefore alternatively $0, -1, 0, -1, \dots, 0$. We get the lower estimate (using the lower bound $\cos^2(3t) \geq 0$ over every interval as before)

$$\begin{aligned} &\int_0^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt \\ &= \int_0^{\pi/6} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt + \dots + \int_{11\pi/6}^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt \\ &\geq \int_0^{\pi/6} \sqrt{25 + 16(0) + 4(0)} dt + \int_{\pi/6}^{3\pi/6} \sqrt{25 + 16(-1) + 4(0)} dt + \dots + \int_{11\pi/6}^{2\pi} \sqrt{25 + 16(0) + 4(0)} dt \\ &= 5 \left(\frac{\pi}{6} \right) + 3 \left(\frac{3\pi}{6} - \frac{\pi}{6} \right) + 5 \left(\frac{5\pi}{6} - \frac{3\pi}{6} \right) + \dots + 5 \left(2\pi - \frac{11\pi}{6} \right) \\ &= 5 \left(\frac{\pi}{6} + \frac{2\pi}{6} + \frac{2\pi}{6} + \frac{\pi}{6} \right) + 3 \left(\frac{2\pi}{6} + \frac{2\pi}{6} + \frac{2\pi}{6} \right) \\ &= 5\pi + 3\pi \\ &= 8\pi. \end{aligned}$$

Similarly, the upper bound on $\cos(3t)$ is alternatively $1, 0, 1, 0, \dots, 1$. We get the upper

estimate (using the bound $\cos^2(3t) \leq 1$ over every interval)

$$\begin{aligned}
& \int_0^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt \\
&= \int_0^{\pi/6} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt + \cdots + \int_{11\pi/6}^{2\pi} \sqrt{25 + 16 \cos(3t) + 4 \cos^2(3t)} dt \\
&\leq \int_0^{\pi/6} \sqrt{25 + 16(1) + 4(1)} dt + \int_{\pi/6}^{3\pi/6} \sqrt{25 + 16(0) + 4(1)} dt + \cdots + \int_{11\pi/6}^{2\pi} \sqrt{25 + 16(1) + 4(1)} dt \\
&= 3\sqrt{5} \left(\frac{\pi}{6} \right) + \sqrt{29} \left(\frac{3\pi}{6} - \frac{\pi}{6} \right) + 3\sqrt{5} \left(\frac{5\pi}{6} - \frac{3\pi}{6} \right) + \cdots + 3\sqrt{5} \left(2\pi - \frac{11\pi}{6} \right) \\
&= 3\sqrt{5} \left(\frac{\pi}{6} + \frac{2\pi}{6} + \frac{2\pi}{6} + \frac{\pi}{6} \right) + \sqrt{29} \left(\frac{2\pi}{6} + \frac{2\pi}{6} + \frac{2\pi}{6} \right) \\
&= (3\sqrt{5} + \sqrt{29})\pi.
\end{aligned}$$