L35: Gauss' Law

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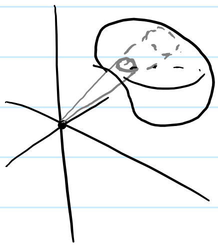
Gauss' Law

$$\vec{F}(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|^3} \quad \text{(inverse square field)}$$

HW12: $\operatorname{div}(\vec{F}) = 0$ for $\vec{x} \neq 0$

$$\iint_{x^2+y^2+z^2=\varepsilon} \vec{F}\cdot d\vec{S} = 4\pi \qquad \text{(oriented outward)}$$

Let S be a simple closed surface.
Let R be the enclosed region and write $\bar{R} = S$ together with R.

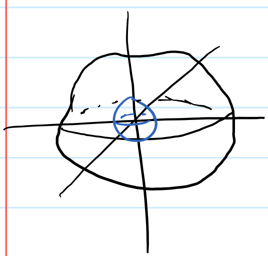


If $0 \notin \bar{R}$, Text then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}\vec{F}\, dV = 0$$

[Ex. $\underbrace{x^2+y^2+z^2<1}_{R}$, $\underbrace{x^2+y^2+z^2=1}_{S}$, $\underbrace{x^2+y^2+z^2\leq 1}_{\bar{R}}$]

Usually do not make a distinction between R and R bar, because the triple integrals of a bounded function over R and R bar are equal (adding a surface with zero area does not change the integral with respect to volume)



If $0 \in R$, there is a small ball of radius $\varepsilon > 0$ around the origin contained in R

By the extended divergence theorem,

$$\iint_S \vec{F}\cdot d\vec{S} - \iint_{\partial B_\varepsilon} \vec{F}\cdot d\vec{S} = \iiint_{\substack{R \text{ with} \\ B_\varepsilon \text{ removed}}} \operatorname{div}(\vec{F})\, dV = 0$$

$$\iint_S \vec{F}\cdot d\vec{S} = 4\pi$$

Conclusion: For any closed simple surface,

$$\iint_S \vec{F}\cdot dS = \begin{cases} 4\pi, & \vec{0} \in R \\ 0, & \vec{0} \notin \bar{R} \end{cases}$$

Application:

Force due to gravity exerted by a mass m at $\vec{x}_0$ on a unit mass at $\vec{x}$

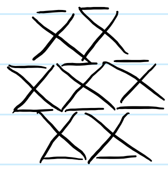
$$\vec{F}_{\vec{x}_0}(\vec{x}) = -Gm\frac{\vec{x}-\vec{x}_0}{\|\vec{x}-\vec{x}_0\|^3}$$

$\vec{x}_0$ $\qquad\qquad\qquad$ $\|\vec{x}-\vec{x}_0\|$

It follows that, for a simple closed surface S,

$$\iint_S \vec{F}_{\vec{x}_0} \cdot d\vec{S} = \begin{cases} -Gm4\pi, & \vec{x}_0 \in R \\ 0, & \vec{x}_0 \notin \bar{R} \end{cases}$$

For a system of finitely many masses, $m_1, \dots, m_n$, at $\vec{x}_1, \dots, \vec{x}_n$ call the force field due to their gravitational pull on a point mass $\vec{G}$.



Any simple closed surface S

$$\iint \vec{G} \cdot d\vec{S} = -4\pi G \sum_{\vec{x}_i \in R} m_i$$

For a continuous distribution of mass, with density δ:

$$\iint_S \vec{G} \cdot d\vec{S} = -4\pi G \iiint_R \delta \, dV$$

(Needs more arguments, but true)

$$\iint_S \vec{G} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{G}) \, dV$$

Because S was arbitrary,

$$\boxed{\operatorname{div}\vec{G} = -4\pi G \delta}$$

local statement of Gauss' Law.

In electrodynamics, $\vec{E}(\vec{x}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$ (Coulomb law)

$$\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$$

density of charge

It is also true that $\vec{G}$ is conservative:

$$\boxed{\vec{G} = \operatorname{grad}\phi}$$

So, $\underbrace{\operatorname{div}(\operatorname{grad}\phi)} = -4\pi G\delta$

Laplacian, denoted Δ or ∇^2

In cartesian coordinates. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

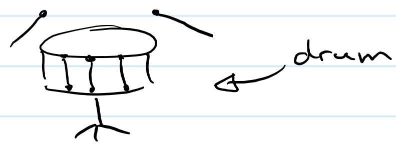
$\Delta\phi = f$ Poisson's Equation

$\Delta \phi = 0$ Laplace's Equation

Describes many things:

- Electrostatic potential $\quad \Delta \phi = \frac{\rho}{\varepsilon_0}$
- Gravitational potential $\quad \Delta \phi = -4\pi G \delta$
- Heat equation $\quad k \Delta T = \partial T / \partial t$

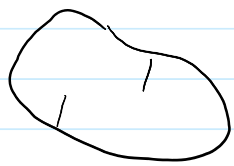
  Diffusion equation ...
- Stretched membranes



Solutions to $\Delta \phi = 0$ are called harmonic functions

Uniqueness property:

If S is a simple closed surface and $\phi_1(\vec{x}) = \phi_2(\vec{x})$ for all $\vec{x} \in S$ and $\Delta \phi_1 = \Delta \phi_2 = 0$, then $\phi_1 = \phi_2$ on the region enclosed by S.



Green's First Identity

f, g continuous second partials

R region (satisfying hypotheses of divergence theorem)

$$\iiint_R \operatorname{grad}(f) \cdot \operatorname{grad}(g) \, dV + \iiint_R f \Delta g \, dV = \iint_S f \operatorname{grad}(g) \cdot d\vec{S}$$

Proof:

First, note that:

$$\operatorname{div}(f\vec{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)$$
$$= \frac{\partial f}{\partial x} F_1 + f \frac{\partial F_1}{\partial x} + \dots + \frac{\partial f}{\partial z} F_3 + f \frac{\partial F_3}{\partial z}$$
$$= \operatorname{grad}(f) \cdot \vec{F} + f \operatorname{div}(\vec{F})$$

Apply this to $\vec{F} = \operatorname{grad}(g)$

$$\operatorname{div}(f \operatorname{grad}(g)) = \operatorname{grad}(f) \cdot \operatorname{grad}(g) + f \Delta g$$

Claim follows by the div. thm. $\blacksquare$

Suppose $\Delta \phi = 0$, with $\phi(\vec{x}) = 0$ on S

Take $f = g = \phi$ in Green's first identity.

$$\iiint_R \|\mathrm{grad}(\phi)\|^2 \, dV + 0 = 0$$

Because $\|\mathrm{grad}(\phi)\|^2 \geq 0$, $\|\mathrm{grad}(\phi)\|^2 = 0$ everywhere.

So $\mathrm{grad}(\phi) = 0$ and $\phi = C$, for some $C \in \mathbb{R}$

But $\phi(\vec{x}) = 0$ on S, so $\phi = 0$.

If $\Delta \phi_1 = \Delta \phi_2$, and $\phi_1(\vec{x}) = \phi_2(\vec{x})$ on S,

Then $\tilde{\phi} = \phi_1 - \phi_2$

Satisfies $\Delta \tilde{\phi} = 0$, and $\tilde{\phi}(\vec{x}) = 0$ on S.

$\Rightarrow \tilde{\phi} = 0$ on R

So $\phi_1 = \phi_2$ throughout R