L34: Extended Versions of FTCs

November 30, 2016 11:33 AM

Announcements:

- HW11, Q5: Need absolute value of det $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

  (or compute det $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$)

Extended Versions of FTCs

FTsC? (Fundamental Theorems of Calculus)

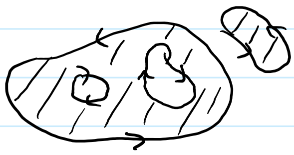
Theorem (Green, Work)

R - region in $\mathbb{R}^2$ (bounded)

∂R = finite union of piecewise curves, oriented so that R appears on the left as one goes around.

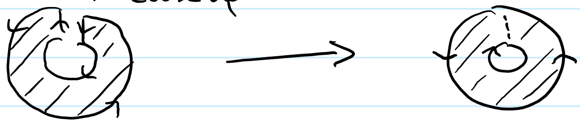
$\vec{F}$ - vector field with continuous partials.

$$\int_{\partial R} \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}_2 \vec{F}\, dA$$



sketch of proof:

→← close up

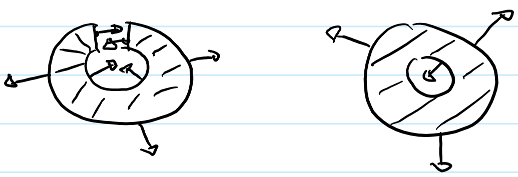


Theorem (Green, Flux)

R, $\vec{F}$ - as before.

∂R oriented with normals away from R.

$$\int_{\partial R} \vec{F}\cdot\hat{n}\, ds = \iint_R \operatorname{div}\vec{F}\, dA$$



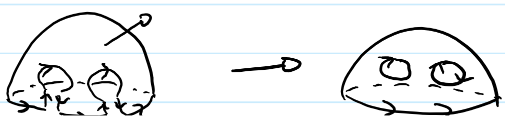
Theorem (Stokes')

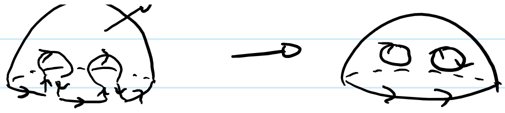
S - surface in $\mathbb{R}^3$ (smooth, orientable, bounded)

∂S - finitely many piecewise curves. ("caveman-induced orientation")

$\vec{F}$ - vector field with continuous first partials

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}_3 \vec{F}\cdot d\vec{S}$$





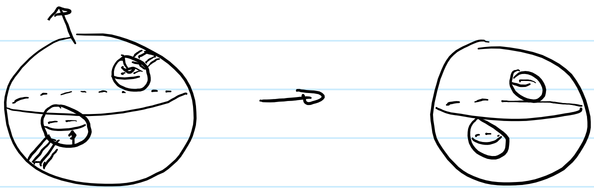
<u>Theorem</u> (Divergence)

R – region in $\mathbb{R}^3$ (bounded, with orientable + smooth boundary)

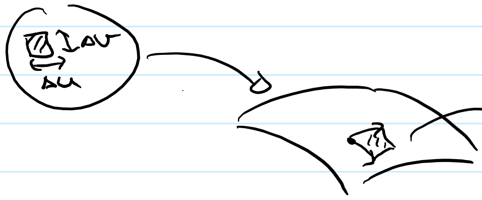
∂R – finite union of surfaces, oriented away from R

$\vec{F}$ - as before

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div} \vec{F}\, dV \qquad \iint_{\partial R} \vec{F} \cdot \hat{N}\, dS$$



$\|\hat{N}\| = \|\vec{T}_u \times \vec{T}_v\| =$ measure of distortion of area by the parametrization.



area is $\approx \|\vec{T}_u \times \vec{T}_v\|\, \Delta u\, \Delta v$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{\sigma}(u,v)) \cdot \vec{N}(u,v)\, du\, dv$$

$$\iint_S \vec{F} \cdot \hat{N}\, dS = \iint_D \vec{F}(\vec{\sigma}(u,v)) \cdot \hat{N}(u,v) \cdot \underbrace{\|\vec{N}(u,v)\|\, du\, dv}_{\text{element of surface area}}$$

$$= \iint_D \vec{F}(\vec{\sigma}(u,v)) \cdot \frac{\vec{N}(u,v)}{\|\vec{N}(u,v)\|} \|\vec{N}(u,v)\|\, du\, dv$$

$$= \iint_D \vec{F}(\vec{\sigma}(u,v)) \cdot \vec{N}(u,v)\, du\, dv$$

$$= \iint_S \vec{F} \cdot d\vec{S}$$

Similarly,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \underbrace{\hat{T}}_{\text{unit tangent}}\, \underbrace{ds}_{\text{arclength element}}$$

<u>Example</u>

$\vec{F} = (x, -y, 2y)$

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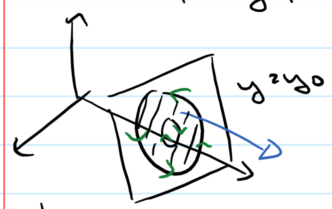
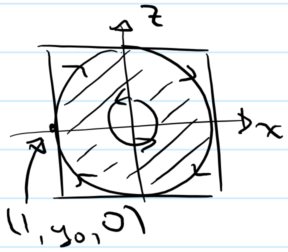


$$\operatorname{curl}\vec{G} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yz - y^2 & -xz & 0 \end{vmatrix}$$

$$\left| \begin{array}{l} 1 \leq x^2 + z^2 \leq 4 \\ y = y_0 \end{array} \right.$$

$$= (x, -y, -2y)$$

$\vec{N} = \vec{e}_y$



$\underline{C_{outer}}$:

$t \to (-2\cos(t), y_0, 2\sin t) \qquad t \in [0, 2\pi]$

$\vec{r}'(t) = (2\sin t, 0, 2\cos t)$

$\vec{G}(\vec{r}(t)) = (-2y_0 \sin(t) - y_0^2, *, 0)$

$\vec{G}(\vec{r}(t)) \cdot \vec{r}'(t) = -4y_0 \sin^2(t) - 2y_0^2 \sin t$

$$\int_{C_{outer}} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} -4y_0 \sin^2(t)\, dt = -4y_0\pi + 0$$

$\underline{C_{inner}}$:

$t \to (\cos t, y_0, \sin t), \quad t \in [0, 2\pi]$

$\vec{r}'(t) = (-\sin t, 0, \cos t)$

$\vec{G}(\vec{r}(t)) = (-y_0 \sin t - y_0^2, *, 0)$

$\vec{G}(\vec{r}(t)) \cdot \vec{r}'(t) = y_0 \sin^2 t + y_0^2 \sin t$

$$\int_{C_{inner}} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} y_0 \sin^2 t + y_0^2 \sin t\, dt$$

$$= y_0\pi + 0$$

$$\int_{\partial S} \vec{G} \cdot d\vec{r} = -4y_0\pi + y_0\pi = 3y_0\pi$$

$\underline{S}$: $(u, v) \to (u\cos(v), y_0, u\sin(v)) \qquad \left| \begin{array}{l} 1 \leq u \leq 2 \\ 0 \leq v \leq 2\pi \end{array} \right.$

$\vec{T}_u = (\cos(v), 0, \sin(v))$

$\vec{T}_v = (-u\sin(v), 0, u\cos(v))$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \cos(v) & 0 & \sin v \end{vmatrix}$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \cos(v) & 0 & \sin v \\ -u\sin(v) & 0 & u\cos(v) \end{vmatrix}$$

$$= (0, u, 0)$$

$$= u\vec{e_y}$$

Actually, this is (0,-u,0), but we should take (0,u,0) for the normal pointing along e_y

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$\vec{F}(\vec{r}(u,v)) = (*, -y_0, *)$$

$$\vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v) = -y_0 u$$

$$\int_0^{2\pi}\int_1^2 -y_0 u \, du\, dv$$

$$= -y_0 \int_0^{2\pi} \left[\frac{u^2}{2}\right]_1^2 dv = -2\pi y_0 \left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= -3\pi y_0$$

$$\boxed{\iint_S \vec{F}\cdot dS = \int_{\partial S} \vec{G}\cdot d\vec{r}}$$