L33: Vector Potential

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Vector Potential

F - vector field with continuous second partials,

It's always true that:

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0$$

In Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

If we're looking for

$$\iint_S \vec{F} \cdot d\vec{S}$$

it is to look for a vector field $\vec{G}$, with

$$\vec{F} = \operatorname{curl} \vec{G}$$

If such $\vec{G}$ exists, then

$$\iint_S \vec{F} \cdot d\vec{S} = \int_C \vec{G} \cdot d\vec{r}$$

$\vec{G}$ is called a vector potential for $\vec{F}$.

If $\operatorname{div} \vec{F}(x,y,z) \neq 0$ for some (x,y,z), $\vec{F}$ can't have a vector potential.

On the other hand, if the domain of $\vec{F}$ has the property that:

| Every closed surface in the domain can be continuously deformed to a point while staying in the domain.

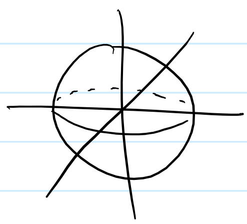
then $\operatorname{div} \vec{F} = 0$ implies $\exists\, \vec{G}$ with $\operatorname{curl} \vec{G} = \vec{F}$.

There exists

Example

$$\vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

Domain: $(x,y,z) \neq (0,0,0)$



HW: $\operatorname{div} \vec{F} = 0$ everywhere on domain,

but

$\iint_S \vec{F} \cdot d\vec{S} = 4\pi$, where S is the unit sphere.

So $\vec{F}$ does not have a vector potential.

$$\iint_S \operatorname{curl}\vec{G}\cdot d\vec{S} \overset{\text{divergence theorem}}{=} \iiint_R \operatorname{div}(\operatorname{curl}\vec{G})\,dV$$
$$= \iiint_R 0\,dV = 0$$

(Non-) Uniqueness

If $\vec{G}$ is a vector potential for $\vec{F}$, and $f: \mathbb{R}^3 \to \mathbb{R}$, then
$$\vec{G} + \operatorname{grad} f$$
is another vector potential for $\vec{F}$.
$$\operatorname{curl}(\vec{G} + \operatorname{grad} f) = \operatorname{curl}\vec{G} + \operatorname{curl}(\operatorname{grad} f)$$
$$= \vec{F} + 0$$
$$= \vec{F}$$

Finding Vector Potentials

Reduction:

Suppose $\vec{G}$ is a vector potential for $\vec{F}$.
$$\vec{G} = (G_1, G_2, G_3)$$

Want: $\frac{\partial f}{\partial z} = -G_3$

So, $f(x,y,z) = \int_{z_0}^{z} -G_3(x,y,t)\,dt$ works.

Then: $\vec{G} + \operatorname{grad} f$ is a vector potential for $\vec{F}$, that looks like $(*, *, 0)$.

For $\vec{G} = (G_1, G_2, 0)$
$$\operatorname{curl}\vec{G} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_1 & G_2 & 0 \end{vmatrix}$$
$$= \left(-\frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)$$

Want to solve:
$$F_1 = -\partial G_2/\partial z$$
$$F_2 = \partial G_1/\partial z$$
$$F_3 = \partial G_2/\partial x - \partial G_1/\partial y$$

$$G_2 = \int_{z_0}^{z} -F_1(x,y,t)\,dt$$

$$G_1 = \int_{z_0}^{z} F_2(x,y,t)\,dt + \underbrace{H_1(x,y)}_{\text{arbitrary function of } x \text{ and } y}$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \int_{z_0}^{z} \left(-\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right)(x,y,t)\,dt - \frac{\partial H_1}{\partial y}$$

Since $\operatorname{div}\vec{F} = 0$,

$$-\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y} = \frac{\partial F_3}{\partial z}$$

$$\int_{z_0}^{z} \frac{\partial F_3}{\partial z}(x,y,t)\,dt = F_3(x,y,z) - F_3(x,y,z_0)$$

So want:

$$\frac{\partial H_1}{\partial y} = -F_3(x,y,z_0)$$

$$H_1 = \int_{y_0}^{y} -F_3(x,u,z_0)\,du \quad \text{works.}$$

Conclusion:

$\vec{F} = (F_1, F_2, F_3)$, then

$$G_1 = \int_{z_0}^{z} F_2(x,y,t)\,dt - \int_{y_0}^{y} F_3(x,u,z_0)\,du$$

$$G_2 = \int_{z_0}^{z} -F_1(x,y,t)\,dt$$

$$G_3 = 0$$

Is one possibility for a vector potential. It can be shown that any other is obtained by adding grad(f) to this one, for some function f

Example

$$\vec{F} = (x, -y, 2y)$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(2y)$$
$$= 1 - 1 + 0 = 0$$

Take $(x_0, y_0, z_0) = (0,0,0)$

$$G_1 = \int_0^z -y\,dt + \int_0^y 2u\,du$$

$$= -yz + [u^2]_0^y = -yz + y^2$$

$$G_2 = \int_0^z -x\,dt = -xz$$

$$G_3 = 0$$

$G_3 = 0$

$$\operatorname{curl}\vec{G} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yz+y^2 & -xz & 0 \end{vmatrix}$$

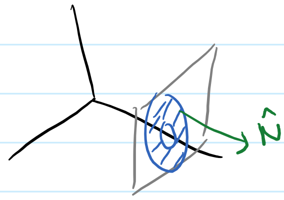
$$= (x, -y, -z-(-z-2y))$$
$$= (x, -y, 2y)$$ the same!

Example

$\vec{F} = (x, -y, 2y)$

$S: \; y = y_0 \;, \; 1 \le x^2 + z^2 \le 2$

$\hat{N} = \vec{e}_y$



Problem: compute $\iint_S \vec{F} \cdot d\vec{S}$