L32: Connections between grad, curl, and div

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Connections between grad, curl, and div

Useful diagram:

$$\left\{\begin{matrix}\text{Functions}\\ \mathbb{R}^3 \to \mathbb{R}\end{matrix}\right\} \xrightarrow[\nabla]{\text{grad}} \left\{\begin{matrix}\text{vector}\\ \text{field}\end{matrix}\right\} \xrightarrow[\nabla\times]{\text{curl}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow[\nabla\cdot]{\text{div}} \left\{\begin{matrix}\text{Functions}\\ \mathbb{R}^3 \to \mathbb{R}\end{matrix}\right\}$$

Slogan: Two in a row = zero — only 3 installments of 29.99\$!

Prop:

$f: \mathbb{R}^3 \to \mathbb{R}$ with continuous second partials

$$\text{curl}(\text{grad} f) = \vec{0}$$

Proof:

$$\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y},\ -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right),\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= (0,0,0) = \vec{0}$$

Prop:

$\vec{F}$ vector field with components having continuous second partials.

$$\text{div}(\text{curl}\,\vec{F}) = 0$$

Proof:

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

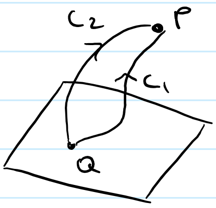
$$= 0$$

$\blacksquare$

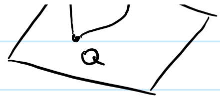
Path Independence (in $\mathbb{R}^3$)

C_1 and C_2 - two paths from Q to P.

$$\int_{C_1} \text{grad} f \cdot d\vec{r} = \int_{C_2} \text{grad} f \cdot d\vec{r}$$



Two Proofs:

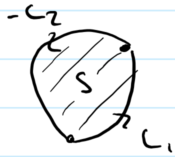


Two Proofs:

1) FTCLI (fundamental theorem of calc for line integrals)

$$\int_{C_1} \text{grad} f \cdot d\vec{r} = f(P) - f(Q)$$
$$= \int_{C_2} \text{grad} f \cdot d\vec{r}$$

2) Let S be a surface with boundary $C_1 - C_2$



Stokes':

$$\int_{C_1 - C_2} \text{grad} f \cdot d\vec{r} = \iint_S \text{curl}(\text{grad} f) \cdot d\vec{S}$$
$$= 0$$

$\boxtimes$

This motivates asking:

Q: Given $\vec{F}$, does there exist $f: \mathbb{R}^3 \to \mathbb{R}$ so that $\vec{F} = \text{grad} f$?

Necessary condition:

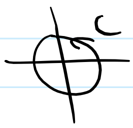
$\text{curl} \vec{F} = 0$ at every point if $\vec{F} = \text{grad} f$.

Sufficient?

Depends on the geometry of the domain of $\vec{F}$!

→ True if domain is simply-connected (every closed curve in the domain can be continuously deformed to a point, staying within the domain).

False in general:



$$\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Defined on $\mathbb{R}^2$ without $(0,0)$

Surface-Independence.

S_1 and S_2 - surfaces with boundary curve C.

$$\iint_{S_1} \text{curl} \vec{F} \cdot d\vec{S}$$
$$= \iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S}$$

If orientations of S_1 and S_2 induce the same orientation on C.

Proofs:

1) $\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S}$

$$= \int_C \vec{F}\cdot d\vec{r}$$

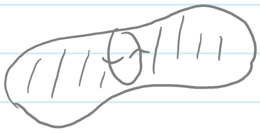
$$= \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$$

2) $\iint_{\pm(S_1 - S_2)} \text{curl}\,\vec{F}\cdot d\vec{S}$

$$= \iiint_R \text{div}(\text{curl}\,\vec{F})\,dV$$

$$= 0$$

assume non-overlapping



Reversing curl:

When is $\vec{F} = \text{curl}\,\vec{G}$ for some vector field $\vec{G}$?
Such a $\vec{G}$ is called vector potential for $\vec{F}$.

Necessary: $\text{div}\,\vec{F} = 0$ (because $\text{div}(\text{curl}\,\vec{G}) = 0$)

Sufficient: When any closed surface in the domain of $\vec{F}$ can be continuously shrunk to a point.

Not sufficient in general.

$\mathbb{R}^3$ without a point.