L31: Stokes' Theorem

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Stokes' Theorem

Defn:

$\vec{F}$ - vector field in $\mathbb{R}^3$

$$\text{curl}\,\vec{F} = \det \begin{bmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \nabla \times \vec{F}$$

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

$$= (\partial F_1/\partial x - \partial F_2/\partial z,\ \partial F_1/\partial z - \partial F_3/\partial x,\ \partial F_2/\partial x - \partial F_1/\partial y)$$

Relation with "$\mathbb{R}^2$ curl":

$\vec{F}(x,y)$ - vector field on $\mathbb{R}^2$

$$(x,y) \Leftrightarrow (x,y,0)$$

$$\vec{G}(x,y,z) = (F_1(x,y), F_2(x,y), 0)$$

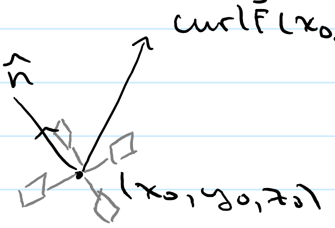
$$\text{curl}\,\vec{G} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= (0, 0, \partial F_2/\partial x - \partial F_1/\partial y)$$

$$\text{curl}_3\,\vec{G} \cdot \vec{e_z} = \text{curl}_2\,\vec{F}$$

Physical Interpretation:

$\vec{F}$ - velocity field of a fluid.

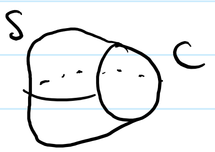


$$\hat{n} \cdot \text{curl}\,\vec{F}(x_0,y_0,z_0)$$
$= 2 \cdot$ angular velocity of paddle wheel with handle along $\hat{n}$

Theorem (Stokes)

S - surface (smooth, bounded, orientable)

C - boundary curve of S

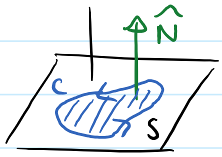


$\vec{F}$ - vector field with continuous 1st derivatives.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S} \qquad \left(\int_{\partial M}\omega = \int_M d\omega\right)$$

Relationship to Green's Theorem:

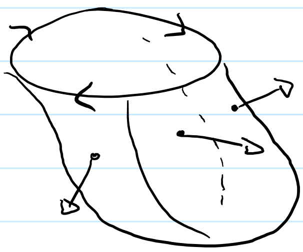
Suppose S is contained in the xy plane.



$$\iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$
$$= \iint_S (\operatorname{curl}\vec{F})\cdot \hat{N}\, dS$$
$$= \iint_S (\operatorname{curl}\vec{F})\cdot \vec{e_z}\, dS$$
$$= \iint_S \left(\partial F_2/\partial x - \partial F_1/\partial y\right) dA$$
$$= \iint_S \operatorname{curl}_z \vec{F}\, dA$$

$$\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ F_1 & F_2 \end{vmatrix}$$

Compatible choice of orientation:

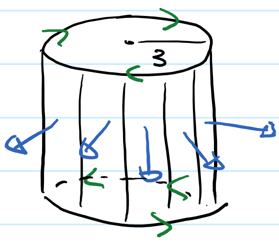


— Orientation — inducing caveman

his name is Bob :)

Example

S: $x^2 + y^2 = 9$, $0 \le z \le 2$ $\qquad \vec{F}(x,y,z) = (2yz, xz, xy)$



$$\partial S = C_1 + C_2$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2yz & xz & xy \end{vmatrix}$$
$$= (x - x, -(y - 2y), z - 2z)$$
$$= (0, y, -z)$$

$$\iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$

$(\theta, v) \to (3\cos\theta, 3\sin\theta, v)$ $\quad \begin{vmatrix} 0 \le \theta \le 2\pi \\ 0 \le v \le 2 \end{vmatrix}$

$\vec{T}_\theta = (-3\sin\theta, 3\cos\theta, 0)$
$\vec{T}_v = (0, 0, 1)$

$$\vec{N} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ -3\cos\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\theta, 3\sin\theta, 0)$$

$\text{curl}\,\vec{F}(\vec{\sigma}(\theta, v)) = (0, 3\sin\theta, -v)$
$\text{curl}\,\vec{F}(\vec{\sigma}(\theta, v)) \cdot \vec{N} = 0 + 9\sin^2(\theta) + 0$

$$\int_0^2 \int_0^{2\pi} 9\sin^2\theta \, d\theta \, dv$$

$$= \int_0^2 9\pi \, dv = 18\pi$$

Top circle:
$t \to (3\cos t, 3\sin t, 2) \qquad 0 \le t \le 2\pi$

$\vec{\sigma}(t) = (3\sin t, 3\cos t, 0)$ ↙ z doesn't matter ↳ will cancel out anyways
$\vec{F}(\vec{\sigma}(t)) = (12\sin t, -6\cos t, *)$
$\vec{F}(\vec{r}(t)) \cdot \vec{\sigma}(t) = 18\sin^2 t - 18\cos^2 t$

$$-18 \int_0^{2\pi} \cos^2 t - \sin^2 t \, dt$$

$$= -18 \int_0^{2\pi} \cos(2t) \, dt$$
$$= -18 \left[-\sin(2t)/2 \right]_0^{2\pi} = 0$$

$$\int_0^{2\pi} 18\sin^2 t \, dt = 18\pi$$

Bottom circle:
$t \to (3\cos t, 3\sin t, 0) \qquad 0 \le t \le 2\pi$

$\vec{\sigma}(t) = (-3\sin t, 3\cos t, 0)$
$\vec{F}(\vec{r}(t)) = (0, 0, *)$
$\vec{F}(\vec{r}(t)) \cdot \vec{\sigma}(t) = 0 + 0 + 0 = 0$

$$\int_{\substack{\text{bottom} \\ \text{circle}}} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = 18\pi + 0 = 18\pi$$

$$\iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S} = 18\pi$$