L30: Divergence Theorem

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Divergence Theorem

(Gauss, Ostrogradskii, ...)

Остроградский

Defn:

$\vec{F}$ - vector field in $\mathbb{R}^3_{(x,y,z)}$

$$\operatorname{div}\vec{F} = \frac{\partial \vec{F}_1}{\partial x} + \frac{\partial \vec{F}_2}{\partial y} + \frac{\partial \vec{F}_3}{\partial z}$$

Another notation:

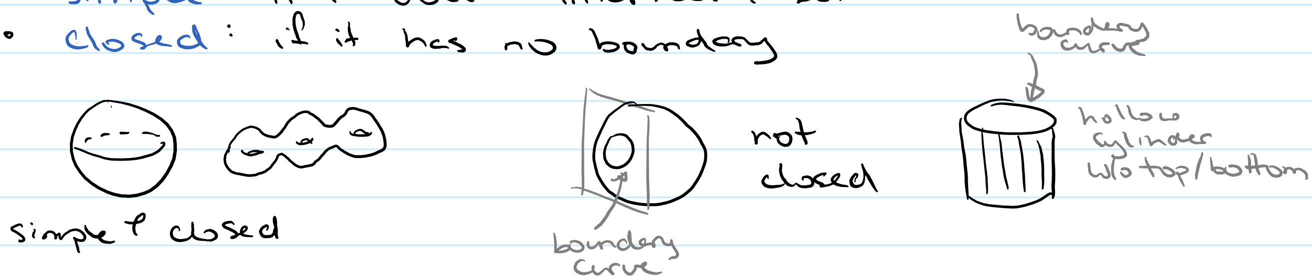
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla \cdot \vec{F} = \operatorname{div}\vec{F}$$

Defn

A surface is called

- simple: if it doesn't intersect itself
- closed: if it has no boundary



Theorem

S - simple closed surface

$\vec{F} = (F_1, F_2, F_3)$ - vector field defined in a neighbourhood of S

(F_1, F_2, F_3 have continuous partial derivatives)

R - solid region bounded by S.

Orient S so that normals point away from R.

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_R \operatorname{div}\vec{F}\, dV$$

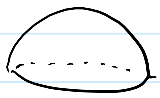
[Analogous to Green for flux:

$$\int_C \hat{F}\cdot\hat{n}\, ds = \iint_R \operatorname{div}\vec{F}\, dA \quad]$$

$$\left(\int_{\partial M} \omega = \int_M d\omega\right)$$

Example

$x^2+y^2+z^2 \le 4$, $\vec{F}(x,y,z) = (0,0,z)$, $z \ge 0$



$\operatorname{div}\vec{F} = 0+0+1 = 1$

$$\iiint_R \operatorname{div}\vec{F}\,dV = \iiint_R 1\,dV = \text{Volume}(R)$$

$$= \frac{4/3 \cdot \pi \cdot 2^3}{2} = \frac{16}{3}\pi$$

$S = S_1 + S_2$

S_1 : top hemisphere

S_2 : disk in xy-plane

$\iint_{S_2} \vec{F}\cdot\vec{dS} = 0$, because $\vec{F} = 0$ on xy-plane.

$S_1 : (\phi, \theta) \to (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$ $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$

$\vec{N} = (4\cos\theta\sin^2\phi, 4\sin\theta\sin^2\phi, 4\cos\phi\sin\phi)$

$\iint_{S_1} \vec{F}\cdot dS$

$$= \int_0^{2\pi}\int_0^{\pi/4} 8\cos^2\phi\sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{8}{3}\cos^3\phi \Big|_0^{\pi/2}\right] d\theta$$

$$= \int_0^{2\pi} 0 - (-8/3 \cdot 1)\, d\theta$$

$$= 8/3 \int_0^{2\pi} d\theta = \frac{16}{3}\pi$$

$$\iint_S \vec{F}\cdot\vec{dS} = \iint_{S_1} + \iint_{S_2} = \frac{16}{3}\pi + 0 = \frac{16}{3}\pi$$

<u>Example</u>

Volume of ball $\underbrace{x^2+y^2+z^2 \le a^2}_{\text{region } R}$

$\vec{F}(x,y,z) = \frac{1}{3}(x,y,z)$

$$\operatorname{div}\vec{F} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

By divergence theorem,

$$\iint_S \vec{F}\cdot dS = \iiint_R 1\,dV = \text{Volume}(R)$$

On the other hand,

$\vec{F}$ is parallel to $\hat{N}$ to every point.

So,

$$\iint_S \vec{F}\cdot \hat{N}\, dS$$

$$= \iint \|\vec{F}\|\, dS$$

$$= \|\vec{F}\| \cdot \iint_S dS$$

$$= \frac{1}{3}\cdot a \cdot (4\pi a^2) = \frac{4}{3}\pi a^3$$

Formulating Continuity Principles

Take fluid flow

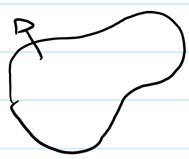
$\delta(x,y,z,t)$ - density

$\vec{v}(x,y,z,t)$ - velocity of the fluid.

$\vec{F}(x,y,z,t) := \delta \cdot \vec{v}$ - rate of mass flow.

Take S - imaginary surface (normals away from bounded R)

Then,



$$\iint_S \vec{F}\cdot d\vec{S}$$

= Rate of flow of mass across S

$$= -\frac{\partial}{\partial t}\iiint_R \delta\, dV$$

Because R doesn't depend on t, can take $\partial/\partial t$ inside.

$$= -\iiint_R \frac{\partial \delta}{\partial t}\, dV$$

On the other hand,

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}\vec{F}\, dV$$

So,

$$\iiint_R \operatorname{div}\vec{F}\, dV = -\iiint_R \frac{\partial \delta}{\partial t}\, dV$$

Lemma

$f : \mathbb{R}^3 \to \mathbb{R}$
$g : \mathbb{R}^3 \to \mathbb{R}$ continuous

If $\iiint_R f\, dV = \iiint_R g\, dV$ for every simple closed R then $f = g$

Proof

Hypothesis is equivalent to $\iiint_R f - g\, dV = 0$ for all R.

If $f \neq g$ there is some point $\vec{x} \in \mathbb{R}^3$ with $|f(x) - \vec{g}(x)| \neq 0$

By continuity, $f(\vec{y}) - g(\vec{y}) \neq 0$ for all $\vec{y}$ in a neighbourhood of $\vec{x}$.
Take some R inside this neighbourhood,
then $\iiint_R f - g \, dV \neq 0$

$\boxtimes$

So, $\operatorname{div} \vec{F} = -\frac{\partial \delta}{\partial t}$

$\left.\operatorname{div} \vec{F} + \frac{\partial \delta}{\partial t} = 0 \right\}$ Conservation of mass (local version)

Can formulate other conservation principles similarily:
- electrical charge
- temperature
- probability density ($\underbrace{|\Psi|^2}_{R}$ wave function)