L29: Spherical Coordinates

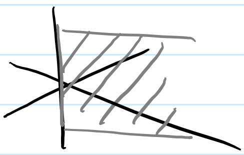
November 17, 2016 1:31 PM

Remember:

If the equation defining a surface in cylindrical coords:

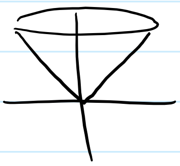
$$z = f(r) \leftarrow \text{no } \theta\text{-dependence}$$

the resulting surface is the surface of revolution of $z = f(x)$ about the z-axis.

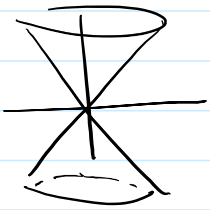


Examples

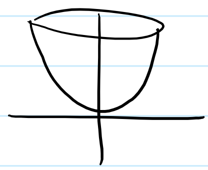
$z = r$



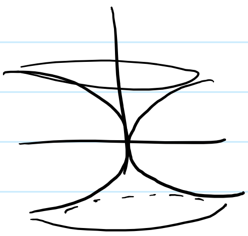
$z^2 = r^2$



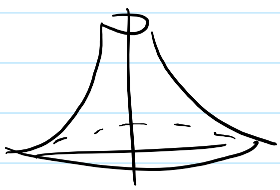
$z = r^2$



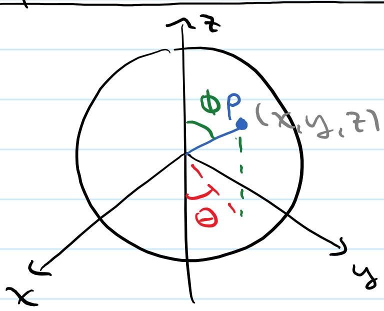
$z^2 = r \Rightarrow z = \sqrt{r}$



$z = \frac{1}{r}$



Spherical Coordinates



(ρ, θ, ϕ)

$\rho \geq 0$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$

$\rho = \sqrt{x^2 + y^2 + z^2}$
$\theta = \arctan\left(\frac{y}{x}\right)$
$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Spherical

Cartesian

$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\cos\phi$
$z = \rho\cos\phi$

$y = \rho \sin\theta \cos\phi$
$z = \rho \cos\phi$

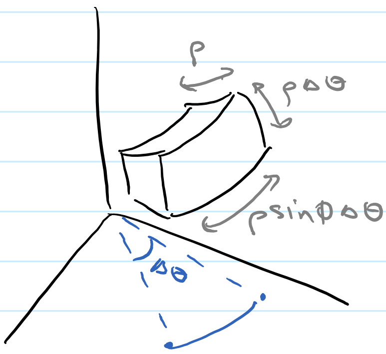
Spherical

$\rho = \sqrt{r^2 + z^2}$
$\theta = \theta$
$\phi = \arctan\left(\frac{r}{z}\right)$

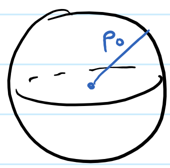
$r = \rho \sin\phi$
$\theta = \theta$
$z = \rho \cos\phi$

Cylindrical
(r, θ, z)

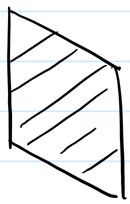
Volume element



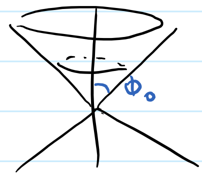
$\rho = \rho_0$



$\theta = \theta_0$

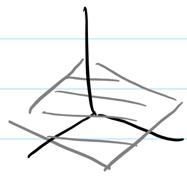


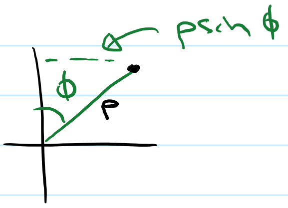
$\phi = \phi_0$



$\Downarrow$

$\phi = \pi/2$

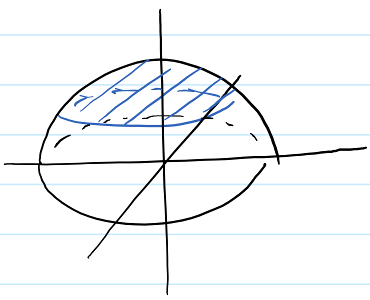


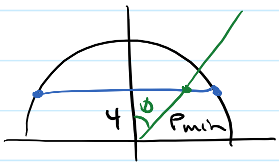


$$\Delta V \approx \rho^2 \sin\phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$$
$$dV = \underbrace{\rho^2 \sin\phi}_{} \, d\rho \, d\phi \, d\theta$$

Example

Find the volume below the sphere $x^2 + y^2 + z^2 = 64$
and the plane $z = 4$



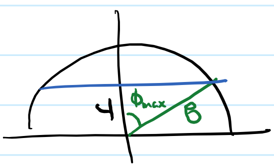


$\frac{4}{\rho_{min}} = \cos\phi$

$\rho_{min} = 4 \sec\phi$

$z = 4 \rightarrow \rho\cos\phi = 4$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{4\sec\phi}^{8} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$\phi_{max} = \arccos\left(\frac{1}{2}\right)$
$= \pi/3$

Change of Variables in Triple Integrals

Idea:

(Inverse Function)

Change of Variables in Triple Integrals

Idea:

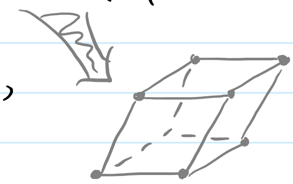
1) A differential map
$$\mathbb{R}^3_{(u,v,w)} \to \mathbb{R}^3_{(x,y,z)}$$
is well approximated by the Jacobian
$$\frac{\partial(x,y,z)}{\partial(u,v,w)}, \text{ as long as } \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$$

(Inverse Function Theorem)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

2) A linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ takes parallelepipeds to parallelepipeds

If $P \subset \mathbb{R}^3_{(u,v,w)}$ is a parallelepided,
$$\mathrm{Vol}(T(P)) = |\det(T)|\,\mathrm{Vol}(P)$$

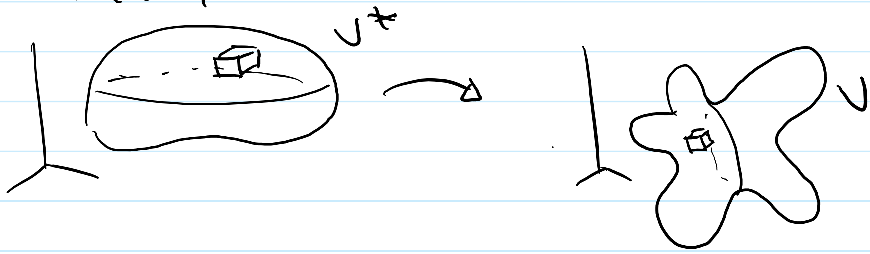


Theorem (Change-of-Variables)

F - differentiable map $\mathbb{R}^3_{(u,v,w)} \to \mathbb{R}^3_{(x,y,z)}$

V^* is a differentiable map $\mathbb{R}^3_{(u,v,w)}$

$V = F(V^*)$



$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{V^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\det\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)\right| du\,dv\,dw$$

Examples

- Cylindrical $(r,\theta,z) \to (r\cos\theta, r\sin\theta, z)$
$$\left|\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$
- Spherical $(\rho,\theta,\phi) \to (\rho\cos\theta\sin\phi, \rho\sin\theta\cos\phi, \rho\cos\phi)$
$$\left|\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi$$