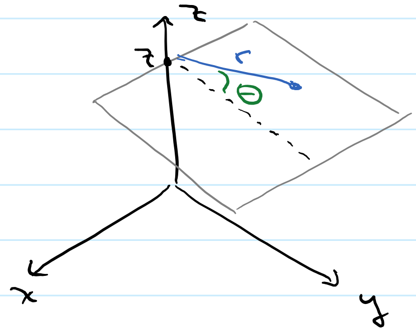
L28: Cylindrical Coordinates

November 16, 2016 11:25 AM

Today: Cylindrical Coordinates

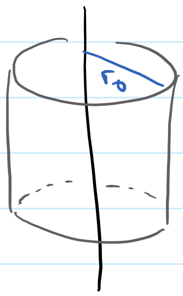


A point's position is described by (r, θ, z)

$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad z \in \mathbb{R}$

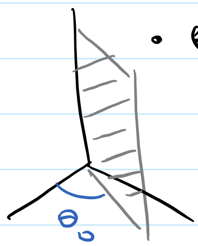
$r = \sqrt{x^2+y^2}$	$x = r\cos\theta$
θ "=" $\arctan\left(\frac{y}{x}\right)$	$y = r\sin\theta$
$z = z$	$z = z$

Common surfaces

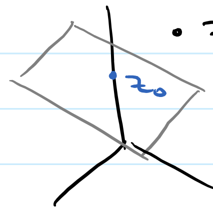


- $r = r_0$

cylinder



- $\theta = \theta_0$



- $z = z_0$

- Sphere

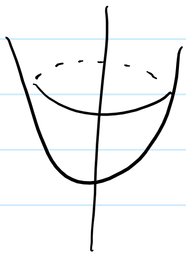
$x^2 + y^2 + z^2 = R^2$

$\hookrightarrow r^2 + z^2 = R^2$

- Paraboloid

$x^2 + y^2 = z$

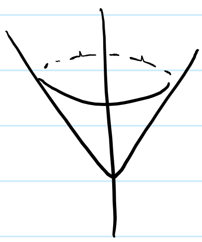
$\hookrightarrow r^2 = z$



- Cone

$x^2 + y^2 = z^2$

$\hookrightarrow r^2 = z^2$

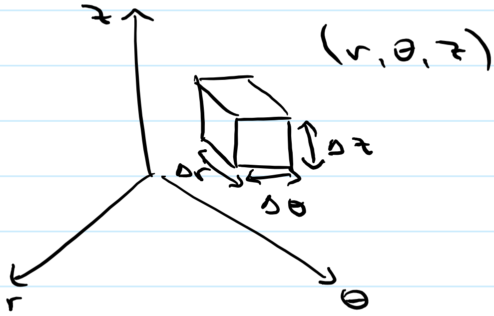


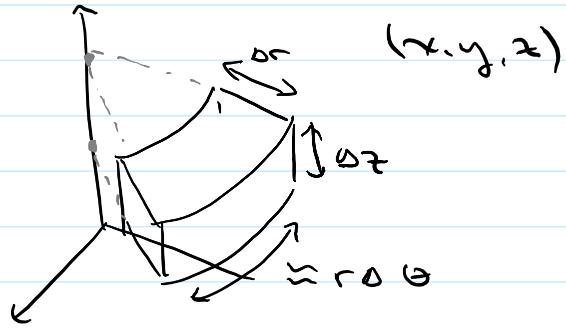
Poorly suited to:

$ax + by + cz = d$

$ar\cos\theta + br\sin\theta + cz = d$

Volume element





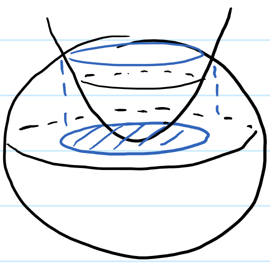
$(r, \theta, z) \to (r\cos\theta, r\sin\theta, z)$

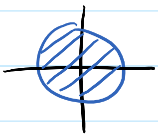
$\Delta V \simeq r \Delta r \Delta\theta \Delta z$

$dV = \underbrace{r}_{\text{Jacobian}} dr\, d\theta\, dz$

Example

Find volume of the region above $x^2+y^2=z$
and below $x^2+y^2+z^2=6$





$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\, dz\, dr\, d\theta$$

$r^2 = z$

$r^2 + z^2 = 6$

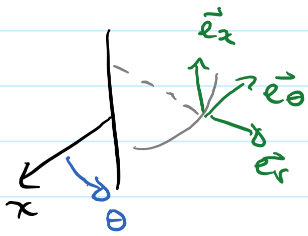
$z^2 + z - 6 = 0$

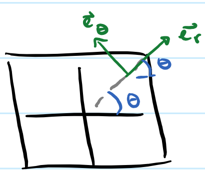
$(z-2)(z+3) = 0$

$z = 2, \quad z = -3$

So $r^2 = z = 2$, or $r = \sqrt{2}$

Direction Vector Fields





$$\vec{e}_r = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$$

$$= \frac{x\vec{e}_x + y\vec{e}_y}{\sqrt{x^2+y^2}}$$

$$\vec{e}_\theta = -\sin\theta\, \vec{e}_x + \cos\theta\, \vec{e}_y$$

$$= \frac{-y\vec{e}_x + x\vec{e}_y}{\sqrt{x^2+y^2}}$$

$$\vec{e}_z = \vec{e}_z$$

Paths:

$t \to (r(t), \theta(t), z(t)), \quad t \in [a,b]$

$$r(t)\, \vec{e}_r(0, \theta(t), z(t)) + z(t)\, \vec{e}_z$$

$$\vec{v}(t) = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \frac{dz}{dt}\vec{e}_z$$

So, if a vector field $\vec{F}$ is expanded as

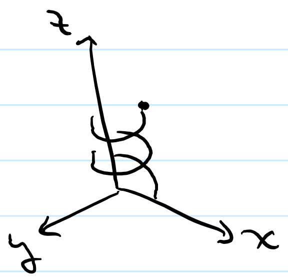
$$\vec{F}(r,\theta,z) = F_r \vec{e_r} + F_\theta \vec{e_\theta} + F_z \vec{e_z}$$

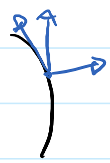
Then work done by $\vec{F}$ along a path is:

$$\int_a^b F_r \frac{dr}{dt} + r F_\theta \frac{d\theta}{dt} + F_z \frac{dz}{dt}\, dt$$

Example

$$t \to (r_0, t, t),\ t \in [0, 4\pi]$$





$$\vec{F} = \frac{1}{r}\vec{e_r} + 2\vec{e_\theta} + 4\vec{e_z}$$

$$\frac{dr}{dt} = 0,\quad \frac{d\theta}{dt} = 1,\quad \frac{dz}{dt} = 1$$

Work:

$$= \int_0^{4\pi} 0 + r_0 \cdot 2 + 4\, dt$$

$$= (8 r_0 + 16)\pi$$

Computing Flux / Surface Area

$$\vec{\sigma}(u,v) \to (r(u,v), \theta(u,v), z(u,v)),\quad (u,v) \in D$$

$\vec{T_u}$ and $\vec{T_v}$ are velocities along coordinate curves.

$$\vec{T_u} = \frac{\partial r}{\partial u}\vec{e_r} + r\frac{\partial \theta}{\partial u}\vec{e_\theta} + \frac{\partial z}{\partial u}\vec{e_z}$$

$$\vec{T_v} = \frac{\partial r}{\partial v}\vec{e_r} + r\frac{\partial \theta}{\partial v}\vec{e_\theta} + \frac{\partial z}{\partial v}\vec{e_z}$$

$$\vec{N} = \det\begin{bmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_z} \\ \frac{\partial r}{\partial u} & \frac{\partial \theta}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial r}{\partial v} & \frac{\partial \theta}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \qquad \|\vec{N}\| = \sqrt{N \cdot N}$$

Integrals w/r/t

Surface Area: $\iint_S f\, dS \qquad f: \mathbb{R}^3 \to \mathbb{R}$

$$= \iint_D f(\vec{\sigma}(u,v))\, \|\vec{N}(u,v)\|\, dv\, du$$

Flux: $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv$