L27: Orientability, Triple Integrals

November 14, 2016 12:27 PM

Equivalent integrals for work & flux

	Geometric	Computational
Work	$\int_C \vec{F}\cdot\hat{T}\,ds$ $= \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{v}(t)}{\lVert\vec{v}(t)\rVert}\lVert v(t)\rVert\,dt$	$\int_C \vec{F}\cdot d\vec{r}$ $= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt$
Flux	$\int_C \vec{F}\cdot\hat{n}\,ds$ $= \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\hat{n}(t)}{\lVert\hat{n}(t)\rVert}\lVert\vec{v}(t)\rVert\,dt$	$= \int_a^b \vec{F}(\vec{r}(t))\cdot\hat{n}_{\pm}(t)\,dt$

Similarily, for flux through a surface,

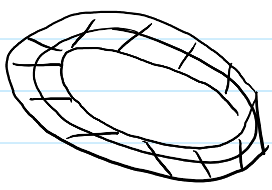
$$\iint_D \vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v)\,dA = \iint_S \vec{F}\cdot d\vec{S}$$

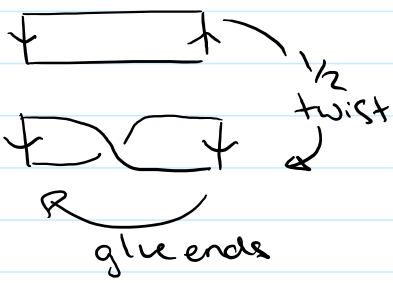
or

$$\iint_D \vec{F}(\vec{r}(u,v))\cdot\frac{\vec{N}(u,v)}{\lVert\vec{N}(u,v)\rVert}\lVert\vec{N}(u,v)\rVert\,dA = \iint_S \vec{F}\cdot\hat{N}\,dS$$

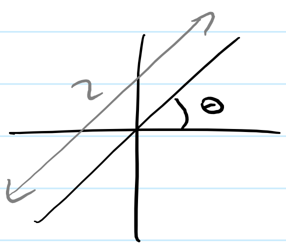
Orientability

Example: Möbius Strip

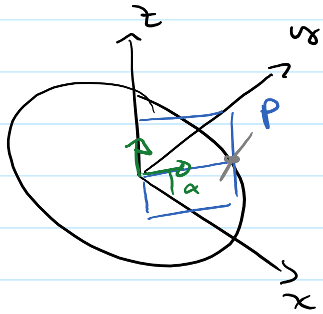




Parametrization:
First, fix some θ



$t \to (t\cos\theta, t\sin\theta)$
$t \in [-1, 1]$



$\cos\alpha \vec{e}_x + \sin\alpha \vec{e}_y$

r = radius of middle circle.

The cross-section of Möbius strip in plane P can be parametrized by: $t \to (t\cos(\frac{\alpha}{2}) + r,\ t\sin(\frac{\alpha}{2}))$ $t \in [-1,1]$

The Möbius strip is then parametrized by:

$$(t,\alpha) \to (t\cos(\tfrac{\alpha}{2}) + r)(\cos\alpha \vec{e}_x + \sin\alpha \vec{e}_y) + t\sin(\tfrac{\alpha}{2})\vec{e}_z$$

$$t \in [-1,1] \quad \alpha \in [0, 2\pi]$$

$$(t,\alpha) \to ((t\cos(\tfrac{\alpha}{2}) + r)\cos\alpha,\ (t\cos(\tfrac{\alpha}{2}) + r)\sin\alpha,\ t\sin(\tfrac{\alpha}{2}))$$

One computes that

$$\vec{N}(0,\alpha) = (r\cos(\alpha)\cos(\tfrac{\alpha}{2}), -r\sin(\alpha)\cos(\tfrac{\alpha}{2}), -r\sin(\tfrac{\alpha}{2})).$$

$$\vec{N}(0,0) = (r, 0, 0)$$
$$\vec{N}(0,2\pi) = (-r, 0, 0)$$

Defn:

A smooth surface is called orientable if one can make a continuous and unique choice of unit normal at every point. Flux only makes sense for orientable surfaces.

Triple Integrals

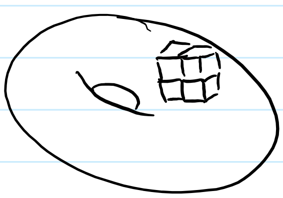
– thrice the fun!

f - continuous function $\mathbb{R}^3 \to \mathbb{R}$

V - volume in $\mathbb{R}^3$

$$\iiint_V f(x,y,z)\,dV$$

$$= \lim_{\mathrm{Vol}(R_{ijk})} \sum_i \sum_j \sum_k f(x_{ijk}, y_{ijk}, z_{ijk})\, \mathrm{Area}(R_{ijk})$$



This measures the "volume" (signed) of the region under the graph of f in $\mathbb{R}^4$

$$w = f(x,y,z)$$

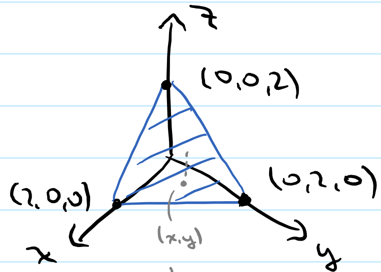
Fubini continues to hold.

Whenever it makes sense,

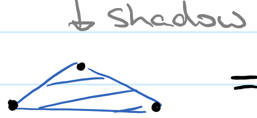
$$\iiint_V f(x,y,z)\,dV = \int_a^b \left(\int_{c(x)}^{d(x)} \left(\int_{e(x,y)}^{h(x,y)} f(x,y,z)\,dz \right) dy \right) dx$$

Examples:

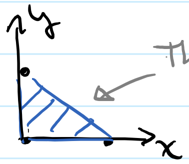
$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x,y,z \geq 0 \\ x+y+z \leq 2 \end{array} \right\}$$



$$\int_0^2 \left(\int_0^{2-x} \left(\int_0^{2-x-y} f(x,y,z)\,dz \right) dy \right) dx$$

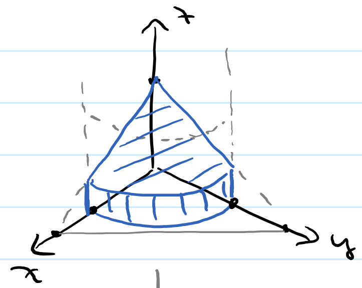


$\Rightarrow$

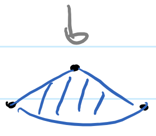


This line is the intersection at $x+y+x=2$ with xy-plane
Equation: $x+y=2$

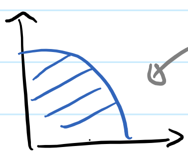
$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x,y,z \geq 0 \\ x^2+y^2 \leq 1 \\ x+y+z \leq \sqrt{2} \end{array} \right\}$$



$$\int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{2}-x-y} f(x,y,z)\,dz \right) dy \right) dx$$



$\Rightarrow$



$x^2+y^2 \leq$