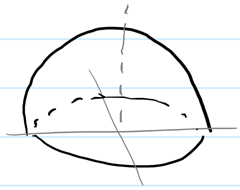
L26: CoM hemisphere, Flux Through a Surface

November 10, 2016 1:30 PM

Exam: Dec 10th 2-5 pm (Sat.) — Smirnov's b-day :)

Centre of mass of a hemisphere:

$$\frac{\iint_S \vec{r}\,\delta\, dS}{\iint_S \delta\, dS} \qquad \delta \text{ - area density}$$



radius a $\quad 0 \le \theta \le 2\pi$

$\delta = 1$ $\quad 0 \le \phi \le \pi/2$

$$(\phi, \theta) \to (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$

$$\vec{N}(\phi,\theta) = \underbrace{c}_{a\sin\phi} \cdot \vec{r}(\phi,\theta)$$

$$= (a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\cos\phi\sin\phi)$$

$$\|\vec{N}\|^2 = a^4\sin^4\phi + a^4\cos^2\phi\sin^2\phi$$

$$= a^4\sin^2\phi$$

$$\|\vec{N}\| = a^2\sin\phi$$

$$\iint_S z\, dS$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \underbrace{a\cos\phi}_{z(\phi,\theta)} \cdot \underbrace{a^2\sin\phi}_{\|\vec{N}(\phi,\theta)\|}\, d\phi\, d\theta$$

$[\sin 2\phi = 2\cos\phi\sin\phi]$

$$= a^3\int_0^{2\pi}\int_0^{\pi/2} \frac{\sin 2\phi}{2}\, d\phi\, d\theta$$

$$= a^3\int_0^{2\pi}\left[-\frac{\cos 2\phi}{4}\right]_{\phi=0}^{\phi=\pi/2} d\theta$$

$$= a^3\int_0^{2\pi}\left[-\frac{1}{4}(\cos(2\cdot\tfrac{\pi}{2}) - \cos(0))\right] d\theta$$

$$= a^3\cdot\frac{1}{2}\int_0^{2\pi} d\theta$$

$$= a^3\cdot\frac{1}{2}\cdot 2\pi = a^3\pi$$

The z-coordinate of C.M. is

$$\frac{a^3\pi}{4\pi a^2/2} = \frac{a}{2}$$

$\iint_S \delta\, dS = \iint_S dS = \text{surf. area}(S)$

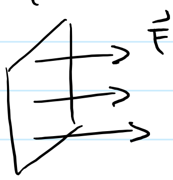
Today: Flux through a surface

Local Picture:

$\vec{F}$ – vector field (interpreted as velocity of fluid)

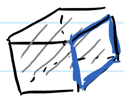
Q: How much fluid passes through a surface per unit time?

Simplest case:



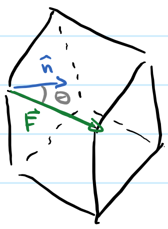
Rectangle R of area A

$\vec{F}$ – perpendicular to the rectangle (and constant)



The amount of fluid that passes through R is $\|\vec{F}\| \cdot \text{Area}(R)$

Skew case:



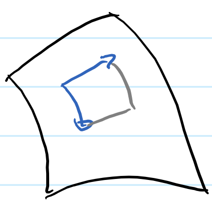
Amount of fluid passing through R/unit time is
$\|\vec{F}\| \, \text{Area}(R) \cdot \cos(\theta)$

If $\hat{n}$ is a unit normal to R, $\vec{F} \cdot \hat{n} \cdot \text{Area}(R)$

Aside:

$\vec{u} \cdot (\vec{v} \times \vec{w})$
= volume of parallelepiped subtended by $\vec{u}, \vec{v}, \vec{w}$.

Global Picture:



For surface S, split into small regions, bounded by coordinate curves.
The flux through each region is approximately:

$$\vec{F} \cdot \frac{\vec{N}}{\|\vec{N}\|} \cdot \|\vec{T}_u \times \vec{T}_v\| = \vec{F} \cdot \vec{N}$$

Defn:

$\vec{F}$ – vector field
S – surface, with continuous choice of normal direction.

Flux of $\vec{F}$ through S is

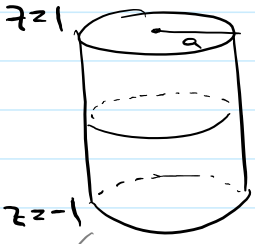
Flux of $\vec{F}$ through S is

$$\iint_S \vec{F}\cdot d\vec{S} := \iint_D \vec{F}(\vec{\sigma}(u,v))\cdot\vec{N}(u,v)\,dA$$

$\hookrightarrow$ Parametrized by $\vec{\sigma}: D \subseteq \mathbb{R}^2_{(u,v)} \to \mathbb{R}^3_{(x,y,z)}$

(Analogous to: $\int_C \vec{F}\cdot\hat{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{n}_{\pm}(t)\,dt$)

Example:



no top or bottom

barrel :)

$(a\cos u,\ a\sin u,\ v) \quad 0 \le u \le 2\pi \quad -1 \le v \le 1$

$\vec{F}(x,y,z) = (x,\ 2y,\ 3z)$

$\vec{T}_u(u,v) = (-a\sin u,\ a\cos u,\ 0)$

$\vec{T}_v(u,v) = (0,\ 0,\ 1)$

$$\vec{T}_u\times\vec{T}_v = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a\cos u,\ a\sin u,\ 0)$

Laplace expansion

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$\|\vec{N}\|^2 = a^2\cos^2 u + a^2\sin^2 u$

$= a^2$

$\|\vec{N}\| = a$

$\vec{F}(\vec{\sigma}(u,v))\cdot\vec{N}(u,v) = (a\cos u,\ 2a\sin u,\ 3v)\cdot(a\cos u,\ a\sin u,\ 0)$

$= a^2\cos^2 u + 2a^2\sin^2 u + 0$

$= a^2 + a^2\sin^2 u$

$\iint_S \vec{F}\cdot d\vec{S} = \int_{-1}^{1}\int_0^{2\pi} a^2 + a^2\sin^2 u\,du\,dv$

Using the fact that $\int_0^{2\pi}\sin^2(u)\,du = \pi$

$= \int_{-1}^{1} 2\pi a^2 + \pi a^2\,dv$

$= 6\pi a^2$