L25: Surface Area, Centre of Mass

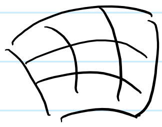
November 9, 2016 11:26 AM

RIP in peace USA :(

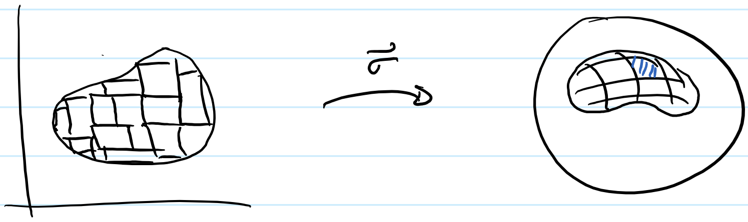
Today: Surface Area
Centre of Mass

Given $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$, want to make sense of $\iint_S f\,dS$, where $S = \vec{\sigma}(D)$ is a parametrized surface.

(Analogue of $\int_C f\,ds$)



Idea: Split the domain D of $\vec{\sigma}$ into rectangles.



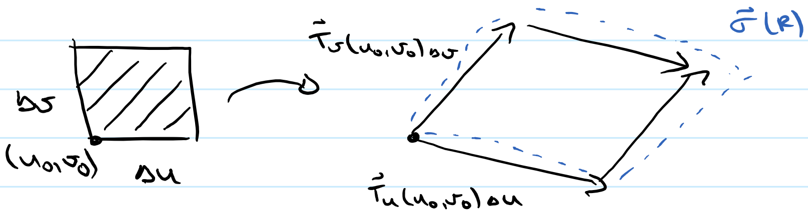
$\iint_S f\,dS$ should be well-approximated by:

$$\lim_{\text{Area}(R_{ij}) \to 0} \sum_i \sum_j f(\sigma(\vec{r}_{ij})) \cdot \text{Area}(\vec{\sigma}(R_{ij}))$$

Linearize $\vec{\sigma}$ near (u_0, v_0) by:

$$\vec{\sigma}(u,v) \simeq \vec{\sigma}(u_0,v_0) + \vec{T}_u(u_0,v_0)(u-u_0) + \vec{T}_v(u_0,v_0)(v-v_0)$$

approximation improves as $u \to u_0$, $v \to v_0$



$$\text{Area}(\vec{\sigma}(R)) \simeq \|\vec{T}_u(u_0,v_0)\Delta u \times \vec{T}_v(u_0,v_0)\Delta v\|$$
$$= \|\vec{T}_u(u_0,v_0) \times \vec{T}_v(u_0,v_0)\|\,\Delta u\,\Delta v$$

$$\lim_{\text{Area}(R_{ij}) \to 0} \sum_i \sum_j f(\vec{\sigma}(\vec{r}_{ij}))\,\|\vec{T}_u(\vec{r}_{ij}) \times \vec{T}_v(\vec{r}_{ij})\| \cdot \text{Area}(R_{ij})$$

$$= \iint_D f\,\|\vec{T}_u(u,v) \times \vec{T}_v(u,v)\|\,du\,dv$$

Defn:

Def^n:

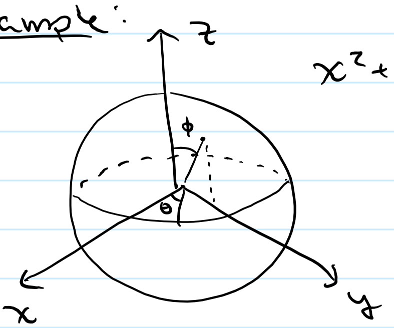
$\vec{\sigma}$ - parametrized surface $D \to \mathbb{R}^3$

f - function $\to \mathbb{R}$ defined in neighbourhood of $\vec{\sigma}(D)$

$$\iint_S f \, dS = \iint_D f(\vec{\sigma}(u,v)) \|\vec{T}_u(u,v) \times \vec{T}_v(u,v)\| \, dA$$

$\hookrightarrow$ analogue of velocity!

Example:



$x^2 + y^2 + z^2 = a^2$

$(\phi, \theta) \to \quad \theta \in [0, 2\pi]$

$\qquad\qquad\quad \phi \in [0, \pi]$

$(a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$

$\vec{T}_\phi(\phi,\theta) = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi)$

$\vec{T}_\theta(\phi,\theta) = (-a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0)$

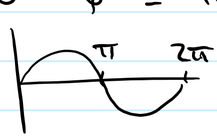
$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$\vec{N}(\phi,\theta) = (a^2\cos\theta\sin^2\phi, \; a^2\sin\theta\sin^2\phi, \; \underbrace{a^2\cos^2\theta\sin\phi\cos\phi - a^2\sin^2\theta\sin\phi\cos\phi}_{= a^2\sin\phi\cos\phi})$$

$$\begin{aligned} \|\vec{N}(\phi,\theta)\|^2 &= a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi \\ &= a^4\sin^2\phi(\sin^2\phi + \cos^2\phi) \\ &= a^4\sin^2\phi \end{aligned}$$

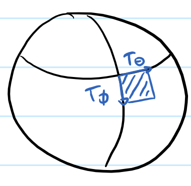
$0 \le \phi \le \pi$

$\|\vec{N}(\phi,\theta)\| = a^2\sin\phi$



$$\int_0^{2\pi}\int_0^{\pi} a^2\sin\phi \, d\phi \, d\theta = \iint_S 1 \, dS$$

$$\begin{aligned} &= a^2 \int_0^{2\pi} [-\cos\phi]_{\phi=0}^{\phi=\pi} \, d\theta \\ &= 2a^2 \int_0^{2\pi} d\theta \\ &= 4\pi a^2 \end{aligned}$$



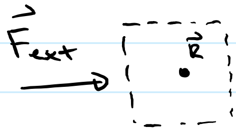
$$\int_0^R 4\pi a^2 \, da$$

$$= \left[\frac{4\pi}{3} a^3\right]^R = \frac{4\pi R^3}{3} = \text{volume of ball of radius } R$$

$$= \left[\frac{4\pi}{3}a^3\right]_0^R = \frac{4\pi R^3}{3} = \text{volume of ball of radius } R$$

Centre of Mass

Motivation: The translational motion of a physical system that is acted on by an external force $\vec{F}_{ext}$ can be described by the motion of a point $\vec{R}$ that $\vec{F}_{ext}$ acts on.



Discrete case: Particles $p_1, \dots, p_n$ with masses $m_1, \dots, m_n$

$\vec{F}_i$ = total force on p_i

$$\vec{F} = \frac{d}{dt}(m_i \vec{v}_i) = \frac{d^2}{dt^2}(m_i \vec{v}_i)$$

$$\sum \vec{F}_i = \vec{F}_{ext}$$

$$\parallel$$

$$\frac{d^2}{dt^2}\left(\sum m_i \vec{r}_i\right)$$

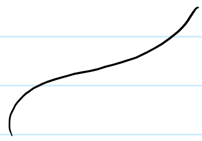
So, if $M = \sum m_i$

$\vec{R} = \frac{1}{M}\sum m_i \vec{r}_i$, then $\vec{F}_{ext} = \frac{d^2}{dt^2}(M\vec{R})$

Continuous cases:

Curve C

Linear density δ

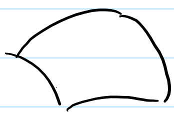


$$\vec{R} = \frac{\int_C \delta \vec{r}\, ds}{\int_C \delta \, ds}$$

$$= \frac{1}{\int_C \delta\, ds}\left(\int_C \delta x\, ds, \int_C \delta y\, ds, \int_C \delta z\, ds\right)$$

Surfaces:

Area density δ



$$\vec{R} = \frac{\iint_S \delta \vec{r}\, dS}{\iint_S \delta \, dS}$$

Volumes:

Volume density δ

$$\vec{R} = \frac{\iiint_V \delta \vec{r}\, dV}{\iiint_V \delta\, dV}$$

$$\vec{R} = \frac{\iiint_V \delta \vec{r}\, dV}{\iiint_V \delta\, dV}$$