L24: Cross Product in R^3, Smoothness, Normals

November 7, 2016 12:30 PM

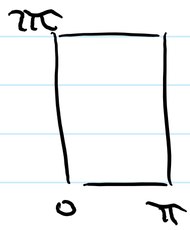
Today: - Cross-Product in $\mathbb{R}^3$
- Smoothness
- Normals

Defn:

A parametrized surface is a map $\vec{\sigma} : D \subseteq \mathbb{R}^2_{(u,v)} \to \mathbb{R}^3_{(x,y,z)}$

A surface described as a graph $z = f(x,y)$ can be parametrized as $(u,v) \to (u, v, f(u,v))$

If u or v are fixed (say $u = u_0$ or $v = v_0$)
then $u \to (u, v_0)$ and
$v \to (u_0, v)$ define curves on $\vec{\sigma}(D)$



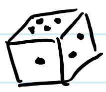
$\to$

These can be thought of as a system of coordinates on $\vec{\sigma}(D)$

Notation:

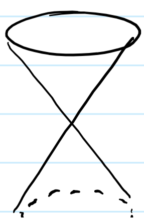
$\vec{T}_u(u_0, v_0)$ - tangent vector to $u \to (u, v_0)$ at (u_0, v_0)
$$= \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right)$$

$\vec{T}_v(u_0, v_0)$ - tangent $v \to (u_0, v)$ at (u_0, v_0)
$$= \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right)$$



Example

$$x^2 + y^2 = z^2$$

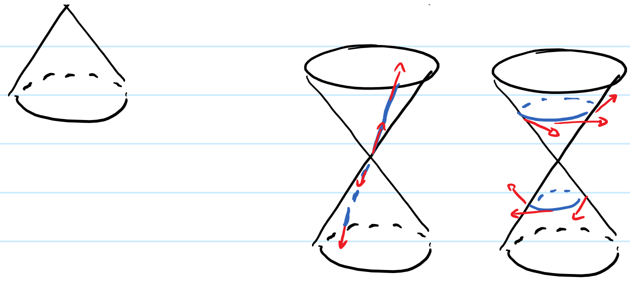


$\to$ circular cone

$(u,v) \mapsto (v\cos(u), v\sin(u), v)$

$v \in \mathbb{R}$ $u \in [0, 2\pi]$

$v \in \mathbb{R}$ $\quad u \in [0, 2\pi]$



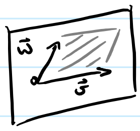
$$\vec{T}_u(u,v) = (-v\sin(u), v\cos(u), 0)$$
$$\vec{T}_v(u,v) = (\cos(u), \sin(u), 1)$$

Defn:
$\vec{\sigma}$ is said to be smooth if $\vec{T}_u(u,v)$ and $\vec{T}_v(u,v)$ span a plane for all $(u,v) \in D$.

Cross Product in $\mathbb{R}^3$:

Defn:
Given $\vec{v} = (x_1, y_1, z_1)$, $\vec{w} = (x_2, y_2, z_2)$
$\vec{v} \times \vec{w}$ is a vector, such that

- $\|\vec{v} \times \vec{w}\|$ = Area of parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.
- Direction determined by "right-hand rule".



$\vec{e}_x \times \vec{e}_x = \vec{0}$	$\vec{e}_x \times \vec{e}_y = \vec{e}_z$	$\vec{e}_x \times \vec{e}_z = -\vec{e}_y$
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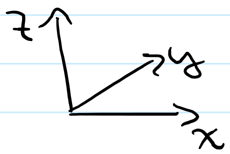
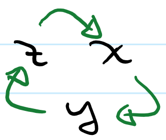




(recycle!)

$$\vec{v} \times \vec{w} = (x_1\vec{e}_x + y_1\vec{e}_y + z_1\vec{e}_z) \times (x_2\vec{e}_x + y_2\vec{e}_y + z_2\vec{e}_z)$$

assuming distributivity in both variables (bilinearity)

$$= 0 + x_1y_2\,\vec{e}_x \times \vec{e}_y + x_1z_2\,\vec{e}_x \times \vec{e}_z + y_1x_2\,\vec{e}_y \times \vec{e}_x$$
$$+ 0 + y_1z_2\,\vec{e}_y \times \vec{e}_z + z_1x_2\,\vec{e}_z \times \vec{e}_x + z_1y_2\,\vec{e}_z \times \vec{e}_y + 0$$
$$= (y_1z_2 - y_2z_1,\ x_2z_1 - x_1z_2,\ x_1y_2 - x_2y_1)$$
$$= \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ x_1 & y_1 & z_1 \\ y_2 & y_2 & z_2 \end{vmatrix}$$

So, $\vec{\sigma}$ is smooth at $\vec{\sigma}(u,v)$ if

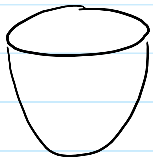
$$\vec{T}_u(u,v) \times \vec{T}_v(u,v) \neq 0$$

In this case, the plane spanned by $\vec{T}_u(u,v)$ and $\vec{T}_v(u,v)$

In this case, the plane spanned by $\vec{T}_u(u,v)$ and $\vec{T}_v(u,v)$ is called the tangent plane to $\vec{\sigma}$ at $\vec{\sigma}(u,v)$ and $\vec{T}_u(u,v) \times \vec{T}_v(u,v) =: \vec{N}(u,v)$ is called the normal to $\vec{\sigma}(u,v)$

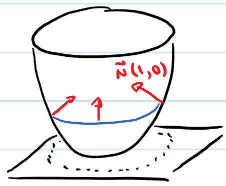
Example

$$(u,v) \to (u, v, u^2+v^2)$$

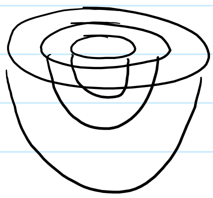


$$\vec{T}_u(u,v) = (1, 0, 2u)$$
$$\vec{T}_v(u,v) = (0, 1, 2v)$$
$$\vec{N}(u,v) = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$
$$\vec{N}(u,v) = (-2u, -2v, 1)$$

e.g. $(u,v) = (1,0)$
$$\vec{N}(1,0) = (-2, 0, 1)$$



If $\vec{\sigma}$ is the level surface of the function $g(x,y,z)$, then ∇g is normal to the surface at every point.



For the function $g(x,y,z) = x^2+y^2-z$
$$\nabla g = (2x, 2y, -1) = -\vec{N}$$

The level surface $g(x,y,z) = 0$ is the paraboloid above, and $\nabla g = -\vec{N}$.