L23: Interpretation of Divergence in Terms of Fluid Flow

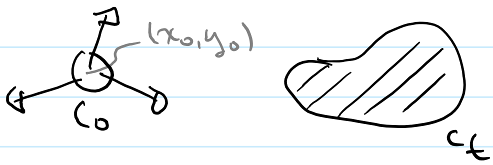
November 3, 2016 1:30 PM

Interpretation of divergence in terms of fluid flow

$\vec{F}$ - vector field, interpreted as velocity of a fluid

C_0 - circle of radius a centered at (x_0, y_0)

C_t - the region that C_0 deforms to by time t.



$$\left.\frac{d\,\text{Area}(C_t)}{dt}\right|_{t=0} = \int_{C_0} \vec{F}\cdot\hat{n}\,dt$$

$$= \iint_{R_0} \operatorname{div}\vec{F}\,dA$$

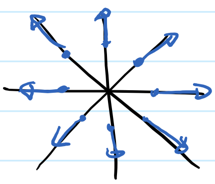
$$\underset{a\text{ small}}{\simeq} \operatorname{div}F(x_0,y_0)\ \text{Area}(C_0)$$

$$\operatorname{div}\vec{F}(x_0,y_0) \simeq \frac{\left.\frac{\text{Area}(C_t)}{dt}\right|_{t=0}}{\text{Area}(C_0)}$$

Intuitively, this is the instantaneous percent change of the area of C_t (at $t=0$)

The approximation becomes exact as $a \to 0$

Example



$\vec{F}(x,y) = (x,y)$

Flow lines are solutions to

$$\left|\begin{array}{ll} x'(t) = x(t) & x(t) = x_0e^t \\ y'(t) = y(t) & y(t) = y_0e^t \end{array}\right.$$

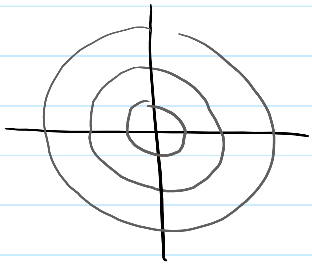
$$\begin{array}{ccc} \vec{v}(t) & = & F(\vec{r}(t)) \\ \parallel & & \parallel \\ (x'(t), y'(t)) & & (F_1(x(t),y(t)), F_2(x(t),y(t))) \end{array}$$

The circle $\theta \to (a\cos\theta, b\sin\theta)$

goes to

$\theta \to (ae^t\cos\theta, be^t\sin\theta)$

C_t = circle of radius ae^t

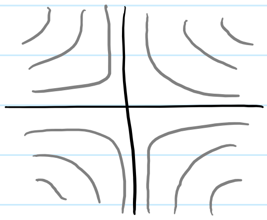


$$\text{Area}(C_t) = \pi(ae^t)^2 = \pi a^2 e^{2t}$$

$$\left.\frac{d\,\text{Area}(C_t)}{dt}\right|_{t=0} = 2\cdot\pi a^2 e^{2t}\Big|_{t=0} = 2\pi a^2$$

$$\frac{\left.\frac{d\,\text{Area}(C_t)}{dt}\right|_{t=0}}{\text{Area}(C_0)} = 2 \qquad \operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$$

Example



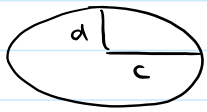
$\vec{F}(x,y) = (x,-y)$

Flow lines are solutions to:

$$\begin{cases} x'(t) = x(t) \\ y'(t) = -y(t) \end{cases} \qquad \begin{aligned} x(t) &= x_0 e^t \\ y(t) &= y_0 e^{-t} \end{aligned}$$

$\theta \to (a\cos\theta, a\sin\theta)$
deforms to
$\theta \to (ae^t\cos\theta, ae^{-t}\sin\theta)$

$$\begin{aligned} \text{Area}(C_t) &= (ae^t)(ae^{-t})\pi \\ &= a^2\pi \\ &= \text{Area}(C_0) \end{aligned}$$



Area = $cd\pi$

$$\frac{\left.\frac{d\,\text{Area}(C_t)}{dt}\right|_{t=0}}{\text{Area}(C_0)} = 0 = \operatorname{div}\vec{F}$$

Terminology

A vector field $\vec{F}$ is called

- incompressible if $\operatorname{div}\vec{F} = 0$ everywhere.
- irrotational if $\operatorname{curl}\vec{F} = 0$ everywhere.

Parameterized Surfaces

Def^n:

A parametrized surface is a differentiable map

$$\sigma : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

"sigma"

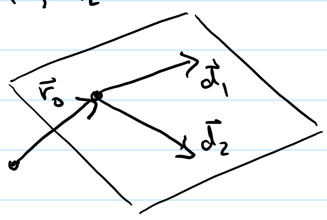
$$(u,v) \to (x(u,v), y(u,v), z(u,v))$$

We will usually require σ to be one-to-one in the interior of D. $\left(\sigma(u,v) = \sigma(u',v') \text{ implies } (u,v) = (u',v') \right)$

The domain D will typically be a rectangle in $\mathbb{R}^2$ (possibly unbounded; open, closed, or neither)

Examples

1. $\vec{r}_0$ - reference point
   $\vec{d}_1, \vec{d}_2$ - directions, not parallel.



$(u,v) \to \vec{r}_0 + u\vec{d}_1 + v\vec{d}_2$

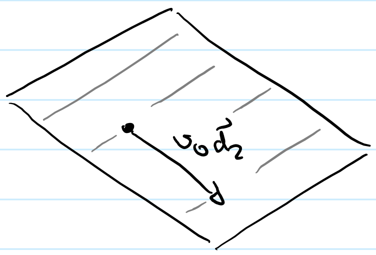
$(u,v) \in \mathbb{R}^3$

is a plane

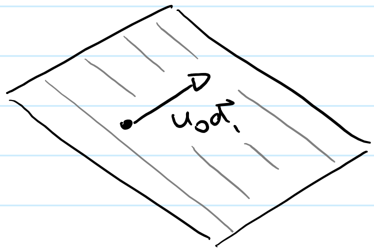
Terminology

Given $\vec{\sigma}$, the path $u \to (u, v_0)$ is called a u-coordinate curve (or simply u-curve) at v_0

Similarily, $v \to (u_0, v)$ is a v-coordinate curve at u_0

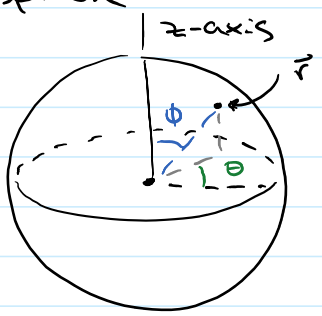


u-curves



v-curves

2. sphere



$0 \leq \theta \leq 2\pi$ polar angle

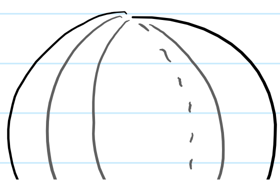
$0 \leq \phi \leq \pi$ azimuthal angle

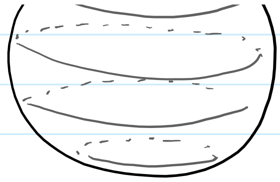
$\vec{r} = a\cos\phi\, \vec{e}_z + a\sin\phi\, \vec{\omega}$

$\vec{\omega} = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$

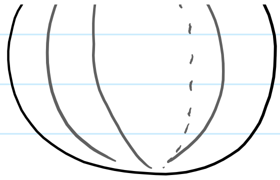
$$\vec{r}(\theta,\phi) = (a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$$

So, $(\theta,\phi) \to \vec{r}(\theta,\phi)$ is a parameterization of a sphere.





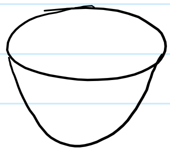
θ-curves



ϕ-curves

3. If $f(x,y): \mathbb{R}^2 \to \mathbb{R}$, its graph is parametrized by
$(u,v) \to (u, v, f(u,v))$

$f(x,y) = x^2 + y^2$



$\Rightarrow$

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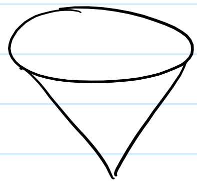
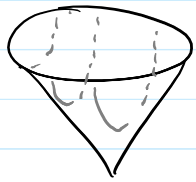
$f(x,y) = \sqrt{x^2+y^2}$



$\Rightarrow$



$\Rightarrow$