# L22: Divergence

November 2, 2016 11:24 AM

## Divergence

Let $\vec{F}$ be a vector field.

Defn:

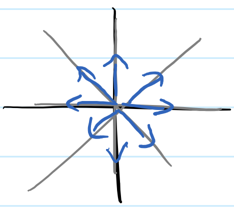
The divergence of $\vec{F}$, $\operatorname{div}\vec{F}$:

$$\operatorname{div}\vec{F}(x,y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \qquad \operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

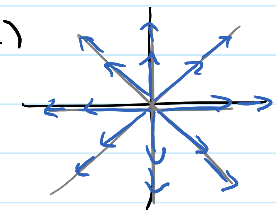
## Examples

1)



$$\vec{F}(x,y) = (x,y)$$
$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$

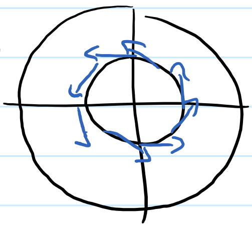
2)



$$\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = \frac{1}{r}\,\vec{e}_r(r,\theta)$$

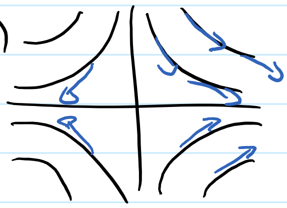
Exercise: check that $\operatorname{div}\vec{F} = 0$ for $(x,y) \neq (0,0)$

3)



$$\vec{F}(x,y) = (-y, x)$$
$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

4)

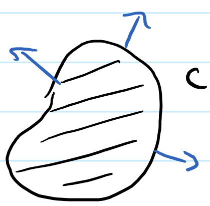


$$\vec{F}(x,y) = (x,-y)$$
$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) = 1 - 1 = 0$$
$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial y}(x) - \frac{\partial}{\partial x}(-y) = 0$$

**Theorem** (Green's Theorem - Flux Version):

C - simple closed curve

R - region bounded by C




Choose normals that point out of R

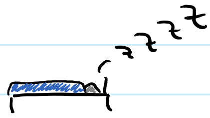
$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$$

Example

C- any simple closed curve $\vec{F}(x,y) = (x,y)$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_R 2\,dA = 2\cdot\text{Area}(R)$$

Planimeter

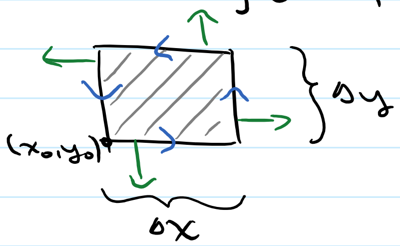




Proof Sketch of Green's Theorem:

Local Picture:

R - rectangle with sides $\Delta x, \Delta y$

C - boundary of R, oriented as in Green's theorem.




$$\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$$

$$F_1(x,y) = F_1(x_0,y_0) + \frac{\partial F_1}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F_1}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ \text{higher order terms.}$$

Similarity for $F_2$ *Taylor series

$$\begin{array}{llll} C_{bottom} & : t \to (x_0+t, y_0) & & 0 \le t \le \Delta x \\ C_{right} & : t \to (x_0+\Delta x, y_0+t) & & 0 \le t \le \Delta y \\ C_{top} & : t \to (x_0+\Delta x - t, y_0+\Delta y) & & 0 \le t \le \Delta x \\ C_{left} & : t \to (x_0, y+\Delta y - t) & & 0 \le t \le \Delta y \end{array}$$

$$\begin{array}{ll|ll} v_{bottom}(t) = & (1,0) & \vec{n}_{bottom}(t) = & (0,-1) \\ v_{right}(t) = & (0,1) & \vec{n}_{right}(t) = & (1,0) \\ v_{top}(t) = & (-1,0) & \vec{n}_{top}(t) = & (0,1) \\ v_{left}(t) = & (0,-1) & \vec{n}_{left}(t) = & (-1,0) \end{array}$$

$$\int_{C_{bottom}} \vec{F}\cdot d\vec{r} + \int_{C_{top}} \vec{F}\cdot d\vec{r}$$

$$= \int_0^{\Delta x} \vec{F}_1(x_0+t, y_0) - \vec{F}_1(x_0+x-t, y_0+\Delta y)\,dt$$

$$\simeq \int_0^{\Delta x} \left(\vec{F}_1(x_0,y_0) + \frac{\partial F_1}{\partial x}(x_0,y_0)t\right) -$$
$$\left(\vec{F}_1(x_0,y_0) + \frac{\partial F_1}{\partial x}(x_0,y_0)(\Delta x - t) + \frac{\partial F_1}{\partial y}(x_0,y_0)\Delta y\right) dt$$

$$= \int_0^{\Delta x} -\frac{\partial F_1}{\partial y}(x_0, y_0)\,\Delta y\, dt$$

$$= -\frac{\partial F}{\partial y}(x_0, y_0)\,\Delta x\,\Delta y$$

$$\int_{C_{left}} \vec{F}\cdot d\vec{r} + \int_{C_{right}} \vec{F}\cdot d\vec{r}$$

$$\approx \frac{\partial F_2}{\partial x}(x_0, y_0)\,\Delta x\,\Delta y$$

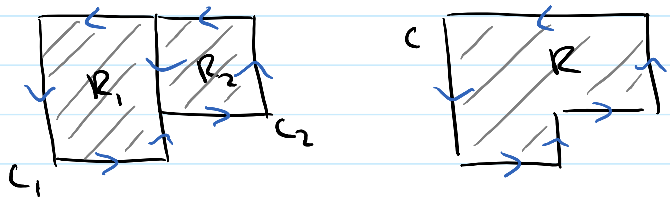
$$\int_C \vec{F}\cdot d\vec{r} = \left(\frac{\partial F_2}{\partial x}(x_0, y_0) - \frac{\partial F_1}{\partial y}(x_0, y_0)\right)\Delta x\,\Delta y$$

$$= \operatorname{curl}\vec{F}(x_0, y_0)\,\Delta x\,\Delta y$$

Approximation becomes better as: $\Delta x \to 0$

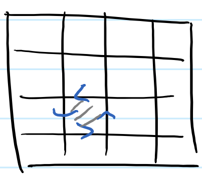
$\Delta y \to 0$

For global region, partition into rectangles:




*Integrals cancel where two boundaries intersect because of opposite orientation.

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$$

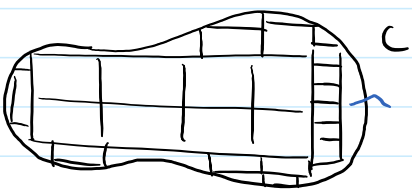


$$\sum\sum \int_{C_i} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$$

In the limit $\Delta x \to 0$

$\Delta y \to 0$

$$\iint_R \operatorname{curl}\vec{F}\,dA = \int_C \vec{F}\cdot d\vec{r}$$




Partition $R$ into rectangles $C_i$

$$\sum_i \int_{C_i} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$$

$$\sum_i \iint_{R_i} \operatorname{curl}\vec{F}\,dA$$

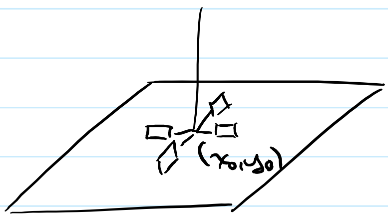
$$\parallel$$

$$\iint_D \operatorname{curl}\vec{F}\,dA$$

## Interpretation of Curl in terms of fluid flow.

Think of $\vec{F}$ as velocity field of some fluid.

Pick $(x_0, y_0)$




Q: How quickly will the paddlewheel rotate?

Terminology:

In this context, $\int_C \vec{F} \cdot d\vec{r}$ is called the circulation.

Depends on the position of the paddles if finitely many
(So imagine taking # paddles to $\infty$)

$$\text{Rotation speed} = \frac{\int_C \vec{F} \cdot d\vec{r}}{\int_C ds}$$

Take $C$ to be a circle of radius $a$

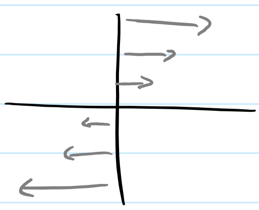
$$= \frac{\iint_R \operatorname{curl} \vec{F} \, dA}{2\pi a}$$

$$\underset{a \text{ small}}{\approx} \frac{\operatorname{curl} \vec{F}(x_0,y_0) \, \pi a^2}{2\pi a}$$

$$\operatorname{curl} \vec{F}(x_0,y_0) \approx \text{Rotational Speed} \, \frac{2}{a}$$

$$= 2 \cdot \text{Angular Speed}$$

$$\operatorname{curl} \vec{F}(x,y) = \lim_{a \to 0} \frac{\int_{\substack{\text{circle of} \\ \text{radius } a \text{ at } (x,y)}} \vec{F} \cdot d\vec{r}}{\pi a^2}$$



$\rightarrow$ a small circle placed anywhere would spin clockwise, so the field has negative curl.