L21: Some Counterexamples

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Today: Some Spooky Counterexamples

1. Fubini might fail

The two integrals over a rectangle might not be equal to eachother.

Take $f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

We have:

$$\frac{\partial}{\partial y} \frac{y}{(x^2+y^2)^2} = \frac{1\cdot(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x} -\frac{x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

So, $\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \left[\frac{y}{x^2+y^2}\right]_{y=0}^{y=1} = \frac{1}{x^2+1}$

$$\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right)dx = \int_0^1 \frac{dx}{x^2+1} = \arctan(1) = \pi/4$$

Integrating w/ x first,

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \left[-\frac{x}{x^2+y^2}\right]_{x=0}^{x=1} = -\frac{1}{y^2+1}$$

So, $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy = -\pi/4$

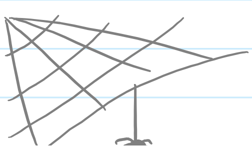
$$\therefore \quad \int_0^1\int_0^1 f(x,y)\,dx\,dy \neq \int_0^1\int_0^1 f(x,y)\,dy\,dx$$

The problem is that $f(x,y) \to \infty$ as $(x,y) \to (0,0)$

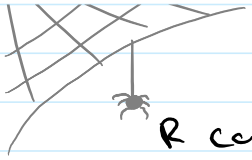
The right hypothesis for Fubini's theorem:
$f(x,y)$ continuous everywhere on R, and there exists $M \in \mathbb{R}$, so that $|f(x,y)| \leq M$ for all $(x,y) \in R$

2. The integral $\iint_R f(x,y)\,dA$ may not exist, but $\int\left(\int f(x,y)\,dx\right)dy = \int\left(\int f(x,y)\,dy\right)dx$

Let $R = \left\{ (x,y) \in [0,1]\times[0,1] : \begin{array}{l} x = \frac{a}{2^n},\ y = \frac{b}{2^n} \\ n \geq 1 \text{ integer} \\ a,b \text{ odd integers} \end{array} \right\}$

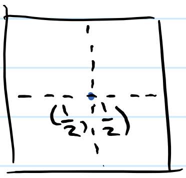


($\qquad\qquad\qquad\qquad$ a, b odd integers)



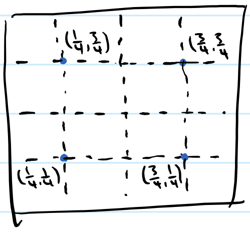
R can be decomposed as a union of the following sets:

R_1:

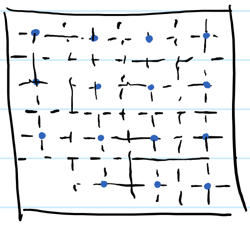


$(\frac{1}{2}, \frac{1}{2})$ is the center of mass of $[0,1]^2 = [0,1] \times [0,1]$

R_2:



R_3:



$R = \bigcup_{k=1}^{\infty} R_k$

$$R_k = \left\{ (x,y) \in [0,1]^2 : \; x = \frac{a}{2^k}, \; y = \frac{b}{2^k} \;\; a, b \text{ odd integers} \right\}$$

$$\chi_R(x,y) = \begin{cases} 1, & (x,y) \in R \\ 0, & (x,y) \notin R \end{cases}$$

"khi"

This is called the characteristic function of R.

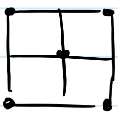
$\iint_{[0,1]^2} \chi_R(x,y)\, dA$ does not exist.

$$\lim_{\text{area}(R_{ij}) \to 0} \sum_i \sum_j \chi_R(x_{ij}, y_{ij}) \cdot \text{Area}(R_{ij}),$$

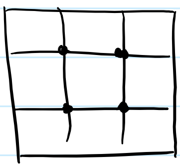
where (x_{ij}, y_{ij}) is any point in R_{ij}.

Take the following sequence of partitions of $[0,1]^2$ into rectangles:

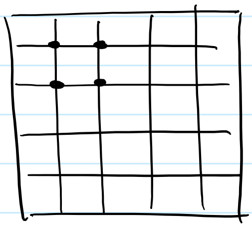
P_1



P_2:



P_3:



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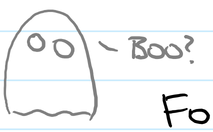

For each P_k, and each R_{ij} one can choose (x_{ij}, y_{ij}) to be either in R or not.

In the first case, $\sum_i \sum_j \chi_R(x_{ij}, y_{ij}) \cdot \text{Area}(R_{ij}) = 1$

second case, $\sum_i \sum_j \chi_R(x_{ij}, y_{ij}) \cdot \text{Area}(R_{ij}) = 0$

So lim over all possible partitions and choices of

(x_{ij}, y_{ij}) can't exist.

On the other hand, both iterated integrals can be checked to exist, and to be equal to zero.

3. Mixed Partials may not be equal

Let $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

At $(0,0)$, need to compute the partials from the definition.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$
$$= 0$$

For $(x,y) \neq (0,0)$

$$f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,y) = \frac{-y^5}{y^4} = -y$$

$$\frac{\partial f}{\partial y}(0,x) = x$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h} = -1$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{h-0}{h} = 1$$

$h \to 0 \qquad h$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}$$

<u>Need</u>: Second partials exist and are continuous.
Then have equality.