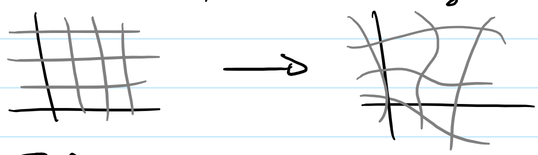
L20: Green's Theorem cont.

October 27, 2016 1:31 PM

Remark:

It is possible to develop the machinery of curvilinear coordinates more completely.

$$T : \mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$$



$$\int_C f\,ds, \quad \int_C \vec{F}\cdot d\vec{r}, \quad \int_C F\cdot\hat{n}\,ds$$

$$\iint_R f\,dA, \quad \vec{e}_u, \quad \vec{e}_v, \quad \vec{v}, \quad \vec{a}$$

$$\nabla f, \quad \nabla\cdot\vec{F}, \quad \nabla\times\vec{F}, \quad (\ldots)$$

Last time

$\vec{F}$ is a vector field

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{pmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Theorem (Green's Theorem – Work Form)

$$\iint_R \text{curl}\,\vec{F}\,dA = \int_C \vec{F}\cdot d\vec{r}$$

where C is the simple closed boundary curve of R, oriented so that R appears on the left (or $\vec{n}_-$ points into R)

To apply to regions bounded by non-simple curves (or even non-connected regions) break up the setup into pieces which are

Also last time, we have seen

$\partial^2 f$ $\partial^2 f$

Also last time, we have seen

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{implies}$$

<u>Prop.</u>

If $\vec{F} = \nabla f$, then curl $\vec{F} = 0$ everywhere

Unfortunately, if $\vec{F}$ is not defined everywhere (and cannot be extended differentiably), it can happen that curl $\vec{F} = 0$ everywhere it is defined but $\vec{F} \neq \nabla f$

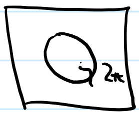
<u>Example</u> (with details left to HW8)

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \text{ for } (x,y) \neq (0,0)$$

curl $\vec{F} = 0$ for $(x,y) \neq (0,0)$

But, for C the unit circle oriented counterclockwise,

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi$$

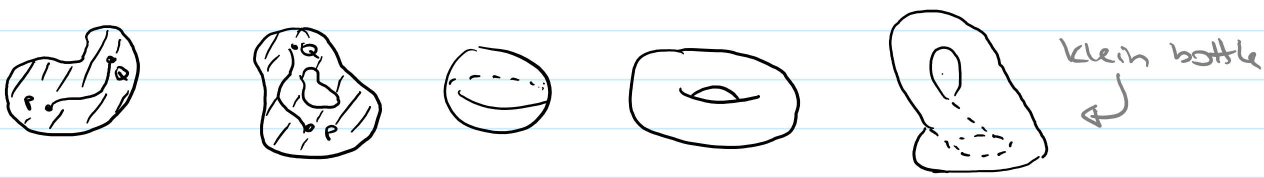


So $\vec{F}$ is not path independent.

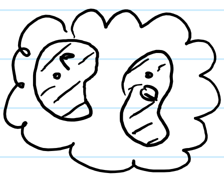
<u>Simply Connected Spaces</u>

<u>Defn:</u>

A space X is path-connected if, for any pair of points P, Q in X, there exists a curve in X connecting P to Q.



are all path connected.



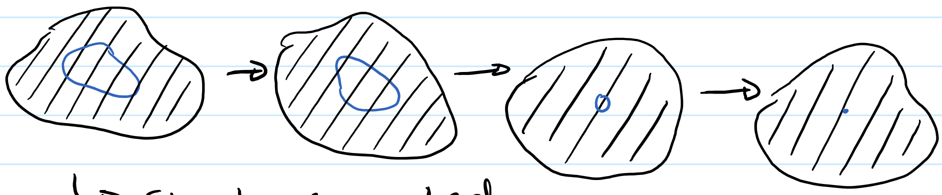
← not path connected.

<u>Defn:</u>

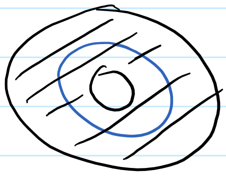
A space X is simply-connected if:

- Path-connected

- Any simple closed curve contained in X can be continuously deformed to a point in X

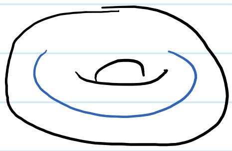


↳ simply connected

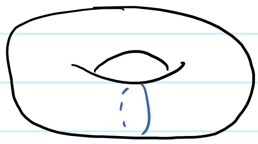


→ not simply connected.

→ simply connected.

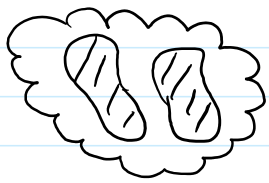


or



– not simply connected

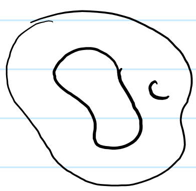
Klein bottle also not simply connected.



– not simply connected, because not path connected.

<u>Remark</u>

If X is simply-connected, and $\vec{F}$ is a vector field defined everywhere in X, then for any simple closed curve C in X, $\vec{F}$ is defined everywhere in the region R bounded by C.



<u>Prop</u>

If $\vec{F}$ is a vector field defined everywhere in a simply-connected space and $\text{curl } \vec{F} = 0$ everywhere in X then $\vec{F}$ is a path independent.

<u>Proof</u>: (Green's Theorem)

Let C be a closed curve in X. Let D be the region bounded

Let C be a closed curve in X. Let R be the region bounded by C. Then $\int_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\, dA$

$$= \iint_R 0\, dA = 0$$

∎

windows

Vista

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\text{F is path independent}} \arrow[r, shift left, "\text{always}"] \arrow[d, shift left, "\text{always}"] & \boxed{\begin{array}{l}\vec{F} = \nabla f \\ f: X \to \mathbb{R}\end{array}} \arrow[l, shift left, "\text{always}"] \arrow[d, "\text{always (if f is good)}"] \\
\boxed{\begin{array}{c}\int_C F\cdot d\vec{r} = 0 \\ C \text{ closed}\end{array}} \arrow[u, shift left, "\text{always}"] & \boxed{\begin{array}{l}\operatorname{curl}\vec{F} = 0 \\ \text{everywhere in } X\end{array}} \arrow[l, "\text{X simply connected}"]
\end{tikzcd}$$

Examples

a) $\vec{F}(x,y) = (x^2y, -2xy)$ defined everywhere on $\mathbb{R}^2$

$$\operatorname{curl}\vec{F} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2y & -2xy \end{pmatrix}$$

$$= -2y - (x^2)$$

$$\neq 0 \text{ at } (x,y) = (1,1), \text{ for instance.}$$

So $\vec{F}$ is not conservative.

b) $\vec{F}(x,y) = (2xy + \cos(2y),\ x^2 - 2x\sin 2y)$ defined everywhere on $\mathbb{R}^2$

$$\operatorname{curl}\vec{F} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy + \cos(2y) & x^2 - 2x\sin 2y \end{pmatrix}$$

$$= 2x - 2\sin(2y) - (2x - 2\sin(2y)) = 0$$

So $\vec{F}$ is conservative.

Indeed, $\vec{F} = \nabla(x^2y + x\cos(2y))$