L19: Change of Vars. Example, Green's Theorem for Work

October 26, 2016 11:28 AM

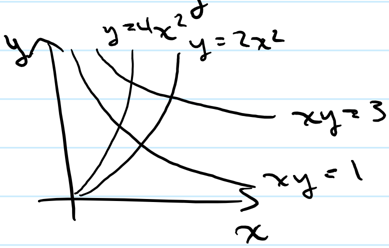
<u>Today</u>: Change of Vars. Example
Green's Theorem for Work

<u>Example</u> Compute double integral

$$f(x,y) = xy^4$$

over the region in $\mathbb{R}^2$ bounded by

$$\begin{cases} xy = 1 \\ xy = 3 \end{cases} \qquad \begin{matrix} y = 2x^2 \\ y = 4x^2 \end{matrix}$$



Want to find $T: \mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$ taking a simple region R^* to R

If: $\begin{cases} u = xy \\ v = y/x^2 \end{cases}$ then can $R^* = [1,3] \times [2,4]$

<u>Try to find inverse</u>:

$$u^2 v = (x^2y^2)(y/x^2) = y^3$$

so, $y = \sqrt[3]{u^2 v}$

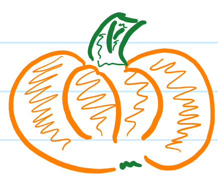
$$\begin{cases} u/v = \dfrac{xy}{y/x^2} = x^3 \\ x = \sqrt[3]{u/v} \end{cases}$$

Take $T: (u,v) \to \left(\sqrt[3]{u/v}, \sqrt[3]{u^2v}\right)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}\left(\frac{u}{v}\right)^{-2/3} \cdot \frac{1}{v} & \frac{1}{3}\left(\frac{u}{v}\right)^{-2/3}\left(\frac{u}{-v^2}\right) \\ \frac{1}{3}(u^2v)^{-2/3} 2uv & \frac{1}{3}(u^2v)^{-2/3} u^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \frac{1}{u^{2/3}v^{1/3}} & -\frac{1}{3}\frac{u^{1/3}}{v^{4/3}} \\ \frac{2}{3}\frac{v^{1/3}}{u^{1/3}} & \frac{1}{3}\frac{u^{2/3}}{v^{2/3}} \end{pmatrix}$$



$$\det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{9}\frac{1}{v} - \left(-\frac{2}{9}\frac{1}{v}\right) = \frac{1}{3v}$$

$\iint \quad \int_1^3 \int_2^4 \quad u^{1/3} \quad 8/2 \quad 4/2 \quad 1$

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \frac{\ }{9}\frac{\ }{v} - \left(-\frac{\ }{9}\frac{\ }{v}\right) = \frac{\ }{3v}$$

$$\iint_R xy^4 \, dA = \int_1^3 \int_2^4 \frac{u^{1/3}}{v^{1/3}} u^{8/3} v^{4/3} \cdot \frac{1}{3v} \, dv \, du$$

$$= \int_1^3 \int_2^4 \frac{u^3}{3} \, dv \, du$$

$$= \int_1^3 \frac{2}{3} u^3 \, du$$

$$= \left[\frac{1}{6} u^4\right]_1^3 = \frac{3^4 - 1}{6} = \frac{80}{6} = \frac{40}{3}$$

Curl:

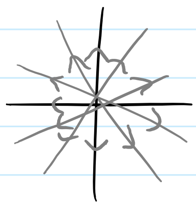
Let $\vec{F} = (F_1(x,y), F_2(x,y))$ be a vector field on $X \subseteq \mathbb{R}^2$.

Def^n:

$$\text{curl } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

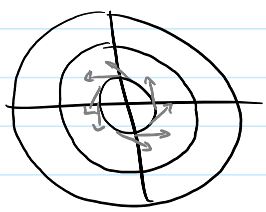
Examples

1. $\vec{F}(x,y) = (x,y)$



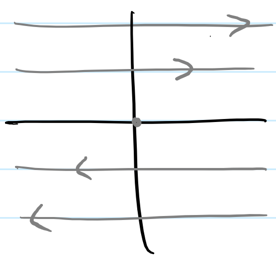
$$\text{curl } \vec{F} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0$$

2. $\vec{F}(x,y) = (-y,x)$



$$\text{curl } \vec{F} = \frac{\partial x}{\partial x} - \left(\frac{\partial(-y)}{\partial y}\right)$$
$$= 1 - (-1) = 2$$

3. $\vec{F}(x,y) = (y,0)$



$$\text{curl } \vec{F} = \frac{\partial 0}{\partial x} - \frac{\partial y}{\partial y} = -1$$

Notation:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

nabla

∇f – gradient of f | $\nabla \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

nabla

∇f – gradient of f
$\left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right)$

$\nabla \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$= \text{curl } \vec{F}$

Theorem (Green's Theorem)
Let R be a region in $\mathbb{R}^2$ bounded by a simple closed curve on C. Orient the curve so that R appears on the left as one traverses C. Let $\vec{F}$ be a vector field defined and differentiable everywhere in R.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA$$

This should be viewed as a two-dimentional version of the fundamental theorem of calculus.
1D: $\int_C \nabla f \cdot d\vec{r} = f(P) - f(Q)$
2D: Green's theorem.

The pattern in both statements is
$$\int_M d\omega = \int_{\partial M} \omega$$
M is some region or curve
∂M is boundary of the region or curve.
ω: object that can be integrated on ∂M
$d\omega$: object obtained from ω by partial differentiation.

Example

$$\vec{F}(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\iint_R \text{curl } \vec{F} \, dA = \int_C \vec{F} \cdot dr = (\sqrt{8} - \sqrt{2})\pi/2$$

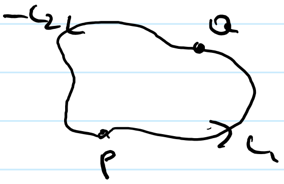
More on Path-Independence
Reminder:
if and only if
$\vec{F}$ is path-independent $\Leftrightarrow$ there exists $\phi : x \to \mathbb{R}$ with $\vec{F} = \nabla \phi$

Prop:

$\vec{F}$ is path-independent if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve C in X.

Proof:

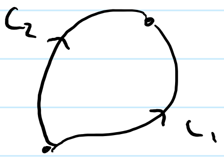
Suppose $\vec{F}$ is path-independent. Let C be a closed curve.



Pick points P, Q (distinct) on C

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = \underline{\underline{0}}$$

Conversely suppose $\int_C \vec{F}\cdot d\vec{r} = 0$ for C closed.
Take P, Q points, path C_1, C_2 from P to Q



Let $C = C_1 - C_2$
Then $0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$

So $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ $\boxtimes$

Suppose $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$

$$= \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\left|\begin{array}{l} \dfrac{\partial F_1}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} \\[2ex] \dfrac{\partial F_2}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y} \end{array}\right.$$

If f is sufficiently good (has continuous second partials)

Then $\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$

So $\operatorname{curl} \vec{F} = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

$$= \frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x} = 0$$

Prop

A conservative vector field has zero curl everywhere.